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SOME RECENT APPLICATIONS OF SEMIRING THEORY

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1. What are semirings anyway?

A semiring is an algebraic structure, consisting of a nonempty set R on which we have defined two operations, **addition** (usually denoted by +) and **multiplication** (usually denoted by $\cdot$ or by concatenation) such that the following conditions hold:

(1) Addition is associative and commutative and has a neutral element. That is to say, $a+(b+c)=(a+b)+c$ and $a+b=b+a$ for all $a,b,c \in R$ and there exists a special element of R, usually denoted by 0, such that $a+0=0+a$ for all $a \in R$. It is very easy to prove that this element is unique.

(2) Multiplication is associative and has a neutral element. That is to say, $a(bc)=(ab)c$ for all $a,b,c \in R$ and there exists a special element of R, usually denoted by 1, such that $a1 = a = 1a$ for all $a \in R$. It is very easy to prove that this element too is unique. In order to avoid trivial cases, we will always assume that $1 \neq 0$, thus insuring that R has at least two distinct elements.

(3) Multiplication distributes over addition from either side. That is to say, $a(b+c)=ab+ac$ and $(a+b)c=ac+bc$ for all $a,b,c \in R$.

(4) The neutral element with respect to addition is multiplicatively absorbing. That is to say, $a0 = 0 = 0a$ for all $a \in R$.

In other words, semirings are just "rings without subtraction". The basic reference for semirings will be [20].

Given a semiring R, a **left *R*-semimodule** M is a nonempty set on which we have operations of addition and scalar multiplication by elements of R (on the left) defined such that:

(1) Addition is associative and commutative and has a neutral element, usually denoted by 0_M. Again, this element can easily be shown to be unique.

(2) For all $a, b \in R$ and $m, m' \in M$, we have $a(bm) = (ab)m$, $(a + b)m = am + bm$, and $a(m + m') = am + am'$.

(3) For all $a \in R$ and $m \in M$ we have $1m = m$ and $0m = 0_M = a0_M$.

For example, it is easy to see that if R is a semiring and if A is a nonempty set, then the set R^A of all functions from A to R is a left R-semimodule, with scalar multiplication and addition being defined elementwise.

Just as the study of rings inevitably involves the study of modules over them, so the study of semirings inevitably involves the study of semimodules over them.

The most trivial example of a semiring which is not a ring is the first algebraic structure we encounter in life: the set of nonnegative integers $\mathbb{N}$, with the usual addition and multiplication. Similarly, the set of nonnegative real numbers $\mathbb{R}_+$ with the usual addition and multiplication is a semiring which is not a ring. The nontrivial examples of semirings first appear in the work of German mathematician Richard Dedekind [11] in 1894, in connection with the algebra of ideals of a commutative ring (one can add and multiply ideals – one cannot subtract them) and were later studied independently by algebraists, especially by the American mathematician H. S. Vandiver, who worked very hard to get them accepted as a fundamental algebraic structure, being basically the “best”structure which includes both rings and bounded distributive lattices [51]. He was not successful, however, and – with only a few exceptions – semirings had fallen into disuse and were well on their way to mathematical oblivion until they were “rescued” during the late 1960’s when real and significant applications were found for them. These include:

Automata theory: During the late 1960’s, Samuel Eilenberg became interested in formal language and automata theory. Basing his ideas on the work of Kleene [27] and working in concert with other major figures in the field, such as Arto Salomaa, Marcel Schützenberger, Jesse Wright, and others, he constructed a comprehensive algebraic theory, published in the first two volumes of a projected (but never completed) four-volume treatise *Automata, Languages, and Machines*, the first volume of which appeared in 1974 [16]. A few years earlier, a slim volume by John Horton Conway [9], based on his own independent study, alamo appeared, and those were followed in 1978 by another seminal work by Salomaa and Soittola [45]. The basic algebraic structures used in these books, and the publications of many other researchers, were semirings. One should note that one of the basic semirings which appears in this corpus is the **tropical semiring** $(\mathbb{N} \cup \{\infty\}, \min, +)$, which was first introduced by Simon [48] (and named by Jean Eric Pin). I mention this semiring in particular, because recently there has been a lot of confusion in the literature between this semiring and the **optimization algebra** $\mathbb{R}_{\min} = (\mathbb{R} \cup \{\infty\}, \min, +)$, which I will mention in a minute. However, we have to keep them straight, because the tropical semiring itself is finding new and important applications. Indeed, the tropical semiring and its properties have been used in recent work to construct efficient algorithms for various classification purposes. A typical example is the work of Allauzen and Mohri in 2003 on testing the twin-primes property [2]. There, they construct an efficient algorithm for testing the twin primes property using weighted automata over commutative and cancellative semirings having complexity $O(|Q|^2 + |E|^2)$, where Q is the set of states of the automaton and E is the set of transitions. This turns out to have practical applications in speech recognition, as well as its theoretical importance.

Optimization theory: The initial impetus comes from a British industrial mathematician, Raymond Cuninghame-Green, who set out his theory in a series of articles, summarized in a lecture notes volume [10] in 1979, whose work is in turn based on the work from the 1960's by an American mathematician B. Giffler. Giffler's research appears to have been mainly military and – to a large extent – unpublished, but he is known in optimization circles for the Giffler-Thomson Scheduling Algorithm, which was published in the 1960's. Other applications studied by Cuninghame-Greene include shortest-path problems and critical-path problems in graph theory, as well as control theory and operations research.

In particular, Cuninghame-Green looked at the following structure: let $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ and define operations $\oplus$ (addition) and $\otimes$ (multiplication) on this set as follows: $a \oplus b = \max\{a, b\}$ and $a \otimes b = a + b$ (where + is the usual addition in $\mathbb{R}$ and $(-\infty) + b = -\infty$ for all $b \in \mathbb{R}_{\max}$. Then $(\mathbb{R}_{\max}, \oplus, \otimes)$ is a semiring, known as the **schedule algebra** or, sometimes, as the **max-plus semiring**. The neutral element of this semiring with respect to addition is $-\infty$ and the neutral element with respect to multiplication is 0. Since every element a in $\mathbb{R}_{\max}$ other than $-\infty$ has an inverse with respect to multiplication, this semiring is in fact a **semifield**, and a surprising amount of linear algebra carries over to "vector spaces"and "matrices"over semifields. The dual of the schedule algebra is the **optimization algebra** $(\mathbb{R}_{\min}, \oplus, \otimes)$, where $\mathbb{R}_{\min} = \mathbb{R} \cup \{\infty\}$, $a \oplus b = \min\{a, b\}$, and

$a \otimes b = a + b$. This too is a semifield, which was successfully applied to optimization problems on graphs by Gondran and Minoux [23] and has become a standard tool in hundreds of papers on optimization. Later, a school of Russian mathematicians, led by academician Victor P. Maslov, was to create a whole new probability theory based on this structure, called **idempotent analysis** (see, for example, [35], [28], [12], [13]), giving interesting applications in quantum physics, which have now become of interest to those computer scientists interested in the problems of quantum computation. In 1994, an important conference on idempotent analysis and its computational aspects was held at Hewlett-Packard's research laboratories in Bristol, England, under the direction of Jeremy Gunawardena [24].

The extensive use of the schedule algebra in the study of discrete-event dynamical systems later centered around a group of French mathematicians at INRIA-Rocquencourt near Paris (who published some of their work under the collective pseudonym Max Plus), and especially in the work of Stéphane Gaubert. Much of their early work was summarized in 1992 in [4]. Complementary work was also done at LIAFA (Laboratoire d'Informatique Algorithmique), an institute of Université Paris VII, under the direction of Jean Eric Pin. The work, both theoretical and applied, of both centers is part of a more general European network of researchers known as ALAPEDES (ALgebraic Approach to Performance Evaluation of Discrete event Systems), coordinated by Geert Jan Olsder. The ALAPEDES website,

http://www.cs.rug.nl/~rein/alapedes/alapedes.html,

also contains links to various publically-available software packages useful for working in this area.

Algebras of formal processes: A third area of application of semirings opened up in the 1980's due to the work of J. A. Bergstra and his collaborators, who defined the notion of an **algebra of communicating processes**, used to formalize the actions in a distributed computing environment. Such an algebra consists of a finite set R of **atomic actions** among which there is a designated action δ (= "deadlock"). On the set R we define two operations, addition (usually called **choice**) and multiplication (usually called **communication merge**) in such a manner that δ is the neutral element with respect to addition and that R, together with these operations, is a semiring. (There is a bit of a problem here with the neutral element with respect to multiplication, but it is easy enough to formally adjoin one if necessary.) There is usually another operation present, called **sequential composition**, which can be considered as an operator acting on this semiring. See, for example, [6]. Other process algebras were later presented by researchers around the world, and a whole area of research in process logic has developed over the past two decades.

In 1969, C. A. R. Hoare introduced a formal system, now known as **Hoare logic**, to investigate specification and verification of well-structured computer programs. By the 1980's, several such systems were introduced, and they in turn led to the definition of various semirings were are used as a context to study program specification and correctness. These include the **dynamic algebras** and **Kleene algebras** studied by Dexter Kozen [29] and David Harel [25], **Hoare algebras** [54], etc.

Generalized fuzzy computation: Bounded distributive lattices are commutative semirings which are both additively idempotent and multiplicatively idempotent. In particular, if $\mathbb{I}$ is the unit interval on the real line, then $(\mathbb{I}, \max, \min)$ is a semiring in which the neutral element with respect to addition is 0 and the neutral element with respect to multiplication is 1. Then, as we have already noted, the set $\mathbb{I}^A$ of functions from A to $\mathbb{I}$ is a semiring, for any nonempty set A. But, following the work of Lofti Zadeh and his hordes of followers, such functions are known as **fuzzy subsets** of A, and so it turns out that semirings are a convenient and useful algebraic framework for fuzzy set theory. Indeed, it was realized quite early, by Joseph Goguen [19] and others, that one should really work with L-fuzzy sets, where L is an arbitrary bounded distributive lattice. This then leads naturally to the consideration of structures of the form R^A, where R is an arbitrary semiring. If one also assumes that A has an algebraic structure, for example if it is a monoid, then we arrive at structures known as **power algebras**, to which I devoted an entire book [21], which I am certainly not in a position to summarize here. Let me just mention that by taking R to be the semiring $\mathbb{N}$ of nonnegative integers, we also get, as a special case, Donald Knuth's theory of **multisets** and by taking R to be the set $\mathbb{N} \cup \{-\infty, \infty\}$ with the usual addition and multiplication suitably augmented, we get the theory of **bags.**

Another important variant on fuzzy computation is to retain $R = \mathbb{I}$ and continue to define addition by $a + b = \max\{a, b\}$, but to change the definition of multiplication. It turns out that operations $\cdot$ satisfying the condition that $(\mathbb{I}, \max, \cdot)$ is a semiring are precisely the **triangular norms** first defined by Menger in connection with probability theory and now finding extensive use in defining interaction rules of economic agents in the theory of fuzzy games. In computer science, triangular norms have been used in everything from image processing to artificial intelligence and various multivalued logics. Dually, the operations $\cdot$ satisfying the condition that $(\mathbb{I}, \min, \cdot)$ is a semiring are called **triangular conorms** and they too have found extensive use, especially in network analysis.

In Zadeh's theory, a function $f \in \mathbb{I}^A$ is an **extent of membership function** used to define a fuzzy subset on a nonempty set A. It is possible to go in another direction as well. Didier Dubois and Henri Prade [15] defined a notion of a **toll subset** of a nonempty set A to be an element of R^A, where R is the semiring $(\mathbb{R}^+ \cup \{\infty\}, \min, \cdot)$. Here, a function $f \in R^A$ is interpreted as a **cost of membership** function. Toll sets have been used extensively to study shortest-path problems and other network routing problems.

The discussion of fuzzy sets lead naturally to the study of fuzzy logic and fuzzy languages, first with values in $\mathbb{I}$ and later with values in an arbitrary semiring. The further transition to fuzzy computation with values in a semiring, with the intention of studying nondeterminism and recursive program schemes, occurred in the work of Wolfgang Wechler [53].

2. Recent results

All the applications mentioned above have been, and continue to be, extensively studied in the literature. However, in recent years additional interesting areas of application have opened up, and it is on those that I wish to concentrate the rest of this talk.

Combinatorial optimization: In 1993, Alexander Barvinok [5] suggested a new approach to combinatorial optimization. To this end, he poses the general problem of combinatorial optimization as follows: given a positive integer n and given a (generally speaking, very large) set S of elements of $\mathbb{N}^n \subset \mathbb{R}^n$ and a vector $v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$, we want to find $t_v = \min\{v \cdot y \mid y \in S\}$, where $\cdot$ is the usual dot product in $\mathbb{R}^n$. Thus, for example, in considering such problems as the Traveling Salesman Problem, n is be the number of edges in the given graph, the set S is the set of all possible paths (where $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in S$ means that there is a path in which, for each $1 \leq h \leq n$, the edge h appears c_h times), and $v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ where a_h is the cost of traversing edge h. What Barvinok the notices is that if we consider this calculation in the optimization algebra $\mathbb{R}_{\min}$, we see that $t_v = p(a_1, \ldots, a_n)$, where

$$p(X_1, \ldots, X_n) = \sum \left\{ X_1^{c_1} \otimes \ldots \otimes X_n^{c_n} \;\middle|\; \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in S \right\}.$$

In other words, we have reduced the problem to the evaluation of a polynomial in several indeterminates over a semifield. This is more than just a notational change, since one can now make use of (appropriately-modified) algorithms for evaluation of polynomials over fields in times often much faster than were available before. For an introduction to this approach, refer to [50]. (It is significant that these fast algorithms are also a result of research in another application of semiring theory, called **path algebras**, especially when it comes to the use of parallel computers. For some interesting results, refer to [49].

Baysian networks and belief propagation: Recently, the problem of calculation of maximum a posteriori log probabilities has led Lior Pachter [41] to consider similar models with applications in statistics. (The logarithmic part, which plays an important role in many applications involving $\mathbb{R}_{\min}$, comes from the existence of a natural isomorphism of semirings $(\mathbb{R}_+, \max, \cdot) \to \mathbb{R}_{\min}$ given by $c \mapsto -\log(c)$; we will come back to this later.) Another application, towards decoding MLSD (= “maximum likelihood sequence detection”) codes, has been considered by Kschischang, Frey, and Loelinger [30]. Indeed, **turbo decoding**, the area in which their work falls, has been explained in terms of Pearl's Belief Propagation Algorithm ([43], [1]), familiar in artificial intelligence under the name of the Junction Tree Algorithm. Also, refer to the seminal work of Shafer and Shenoy on probability propagation [46] through local computation of probability distributions. In Israel, work in this area is being done primarily by Yair Weiss of the Hebrew University [17], [55]. The basic idea in all of this is to somehow exploit independence relations induced by evidence to construct efficient algorithms for probabilistic inference in Baysian networks.

This work provides a framework for a general family of algorithms known as "local message passing algorithms", which have enjoyed considerable interest recently. The most important application of the Belief Propagation Algorithm is to the so-called "hidden Markov chain inference problem", which is also the problem addressed by Pachter. Refer also to [33] and [42]. Similarly, the use of semiring methods to find solutions to deterministic Markov decision processes in polynomial time was considered by Littman in his 1996 thesis [31].

There are many applications of this material in digital signal processing and the construction of sensor networks. For applications of this work in constructing statistical language models, important in text-processing and speech-processing applications, refer to the work of Allauzen, Mohri and Roark [3] at AT&T Labs.

Algebraic geometry over the optimization algebra: As a result of the work of Barvinok, it became interesting to look at the geometric structure of $\mathbb{R}^n_{\min}$. The foremost workers in this area are Mike Develin, Grigory Mikhalkin, and Bernd Sturmfels. See, for example, [14], [37], and [44]. Much of their work is based on results by members of the INRIA discrete-event dynamical systems group [8], by the Russian idempotent-analysis group [32], and by algebraic geometers such as Oleg Viro [52]. One of their major interests is in convex polytopes in the space $\mathbb{R}^n_{\min}$. Here, a subset S

of $\mathbb{R}^n_{\min}$ is **convex** if and only if $a \otimes v \oplus b \otimes w \in S$ whenever $a, b \in \mathbb{R}_{\min}$ and $v, w \in \mathbb{R}^n_{\min}$ (here $\mathbb{R}^n_{\min}$ is considered as a semimodule over $\mathbb{R}_{\min}$). The **convex hull** of a nonempty subset D of $\mathbb{R}^n_{\min}$ consists of all linear combinations $\sum_{i=1}^n a_i \otimes v_i$ of elements of D; in particular a **convex polytope** is the convex hull of a finite subset D of $\mathbb{R}^n_{\min}$.

Many of the standard geometric theorems hold true in this setting. For example,

Theorem (Carathéodory's Theorem over $\mathbb{R}_{\min}$): *If $\varnothing \neq D \subseteq \mathbb{R}^n_{\min}$ is the convex hull of a set of r points, then it is the convex hull of at most n of them.*

Theorem: *Any convex polytope in $\mathbb{R}^n_{\min}$ is the convex hull of a unique minimal set.*

Moreover, the geometric structure in $\mathbb{R}^n_{\min}$ can be used to prove results in "ordinary"algebraic geometry. For example, Mikhalkin [38] established a formula for enumeration of curves of arbitrary genus in toric surfaces by moving over to $\mathbb{R}^n_{\min}$.

There is a problem, of course, of finding appropriate definitions. Consider, for example, trying to define the notion of a linear subspace of $\mathbb{R}^n_{\min}$. The most straightforward definition of a **linear subspace** of $\mathbb{R}^n_{\min}$ is that it is a set consisting of all solutions $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ of a finite set of equations of the form $y \cdot v = y' \cdot v$ for $y, y' \in \mathbb{R}^n_{\min}$. A linear subspace defined by one such equation is a **hyperplane**.

Theorem (Farkas' Lemma over $\mathbb{R}^n_{\min}$): *If $v \in \mathbb{R}^n_{\min}$ and if A is a polytope in $\mathbb{R}^n_{\min}$, then either $v \in A$ or v is separated from A by a hyperplane.*

Linear subspaces of $\mathbb{R}^n_{\min}$ are surely convex. Note that the function $v \mapsto y \cdot v$ is just a homomorphism of $\mathbb{R}_{\min}$-semimodules from

$\mathbb{R}^n_{\min}$ to $\mathbb{R}_{\min}$, so finding the linear subspaces of $\mathbb{R}^n_{\min}$ is just a special case of finding solutions of finite systems of equations of the form $m\alpha = m\beta$, where m is an element of a left R-semimodule M (for some semiring R) and $\alpha, \beta \in Hom(M,N)$ for some left R-semimodule N. The problem, in this generality, is dealt with in my recent book [22].

An alternative method would be to define a linear subspace of $\mathbb{R}^n_{\min}$ to be the convex hull of a finite nonempty set of points of $\mathbb{R}^n_{\min}$. These two definitions do not lead to the same spaces; see [44]. Both of these definitions have problems, however, when it comes to doing algebraic geometry and, indeed, as we move on to more complex geometric objects, other approaches have emerged which better fit the geometric context, albeit often at the cost of algebraic simplicity.

Indeed, consider the following definition of an algebraic variety in $\mathbb{R}^n_{\min}$, put forth by Sturmfels: the **order** of a rational function in one complex variable is the order of its zero or pole at the origin, namely the smallest exponent in the numerator polynomial minus the smallest exponent in the denominator polynomial. This definition of order extends uniquely to the algebraic closure K of the field $\mathbb{C}(t)$ of rational functions, since every nonzero algebraic function $p(t) \in K$ can be locally presented as a **Puiseux series** $p(t) = \sum_{i=1}^{\infty} c_i t^{q(i)}$, where the c_i are nonzero complex numbers and $q(1) < q(2) < \ldots$ are rational numbers with bounded denominators. The **order** of $p(t)$ is then $q(1)$. Similarly, the **order** of $\begin{bmatrix} p_1(t) \\ \vdots \\ p_n(t) \end{bmatrix}$ is $\begin{bmatrix} q_{11} \\ \vdots \\ q_{n1} \end{bmatrix}$, where each q_{h1} is the order of $p_h(t)$. Thus we have a function

$$order : (K \smallsetminus \{0\})^n \to \mathbb{Q}^n \subseteq \mathbb{R}^n.$$

If I is any ideal in the Laurent polynomial ring $K\left[X_1^{\pm 1}, \ldots, X_n^{\pm 1}\right]$, then I defines an affine variety $V(I) \subseteq (K \smallsetminus \{0\})^n$ and its image under *order* is a subset of $\mathbb{Q}^n$. The topological closure $T(I)$ of this image will be an **algebraic variety** in $\mathbb{R}^n_{\min}$. In particular, if the ideal I is generated by linear forms of the form $\sum_{i=1}^n p_i(t)X_i$, then the algebraic variety defined by $V(I)$ is a **Sturmfels linear subspace** of $\mathbb{R}^n_{\min}$.

We should note that such varieties show up in many different contexts of recent research, under various names, among them "logarithmic limit sets", "Bergman fans", "Bieri-Groves sets", and "non-archimedian amoebas". They are strongly connected to the notion of Maslov dequantization, which we will talk about shortly.

The above definition of algebraic varieties in $\mathbb{R}^n_{\min}$ seems, of course, rather artificial and certainly far from the original context with which we started. However, it offers the advantage of actually allowing for a computational procedure for computing such varieties, which I will not go into here.

Moreover, it is possible to give a more intuitive approach. A **monomial** over $\mathbb{R}_{\min}$ is an expression of the form $a \otimes X_1^{c_1} \otimes \ldots \otimes X_n^{c_n}$, where $a \in \mathbb{R}_{\min}$ and where the c_i are nonnegative integers. Thus, each such monomial represents an affine function from $\mathbb{R}^n$ to $\mathbb{R}$ given by

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mapsto a + \sum_{i=1}^{n} c_i x_i .$$

A **polynomial** g over $\mathbb{R}_{\min}$ is a finite sum of monomials

$$\bigoplus_{j=1}^{n} \left[a_j \otimes X_1^{c_{1j}} \otimes \ldots \otimes X_n^{c_{nj}} \right] .$$

Such a polynomial represents a function from $\mathbb{R}^n$ to $\mathbb{R}$ which is piecewise linear and concave (since it is the minimum of a finite number of algebraic functions). Another way of looking at this function is that it is the Legendre transform of the function $j \mapsto -a_j$ (this function is only defined on a finite set of points, but its Legendre transform is defined on all of $\mathbb{R}^n$). The **hypersurface** $T(g)$ determined by such a polynomial is the set of all points $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ at which this function is not linear, namely the set of points x at which the minimum is attained by two or more of the affine functions determined by the monomials. One can then show:

Theorem: *If g is a polynomial over $\mathbb{R}_{\min}$ then there exists a polynomial $f \in K[X_1, \ldots, X_n]$ such that $T(g) = T(I)$, where I is the ideal generated by f.*

An intersection of finitely-many hypersurfaces over $\mathbb{R}_{\min}$ is a **prevariety** over $\mathbb{R}_{\min}$. Every algebraic variety, as previously defined, is a prevariety. The converse, however, is false.

A comprehensive introduction to this theory is given in [44], and it is not my intention to pursue it further. The authors manage to develop a version of Bezout's Theorem for their geometry over $\mathbb{R}_{\min}$, and go into detailed problems of construction, including computational considerations.

Dequantization and amoebas: In the 1980's, as we already noted, Maslov, in his work on optimization, defined the notion of **dequantization** of the semiring $(\mathbb{R}_+, +, \cdot)$ of nonnegative real numbers [34]. For each nonnegative real number h, he considered the semiring $R_h = (\mathbb{R}_+, \oplus_h, \cdot)$, where the operation $\oplus_h$ is defined by:

$$a \oplus_h b = \begin{cases} \max\{a, b\} & \text{if } h = 0 \\ \left[a^{1/h} + b^{1/h}\right]^h & \text{if } h > 0 \end{cases}.$$

Notice that $\max\{a, b\} = \lim_{h \to 0} (a \oplus_h b)$. Also, we note that $R_1 = (\mathbb{R}_+, +, \cdot)$. If $h > 0$, then the semiring R_1 is isomorphic to R_h under the map $a \mapsto a^h$. On the other hand, R_1 is not isomorphic to R_0, since addition in R_0 is idempotent, whereas addition in R_1 is not.

If $1 < t \leq \infty$, then we also have a semiring $R_{[t]} = (\mathbb{R}, \oplus_t, +)$, where

$$a \oplus_t b = \begin{cases} \log_t(t^a + t^b) & \text{if } t < \infty \\ \max\{a, b\} & \text{if } t = \infty \end{cases} .$$

Then the map $\log : R_h \to R_{[t]}$, where $t = e^{1/h}$, is always an isomorphism (here we use the notational convention that $\infty = 1/0$). As it turns out, these observations are connected with a technique of Oleg Viro in algebraic geometry known as **patchworking**, which is in turn related to the above geometric constructions over $\mathbb{R}_{\min}$. The connection between them has been worked out by Mikhalkin, in his theory of **amoebas** [36]. Also, in this connection, refer to recent work of Shustin [47].

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