Left ideals in 1-primitive near-rings

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Near-rings

Definition 1. *A near-ring is a set N together with two binary operations "$+$" and "$\cdot$" such that*

(1) $(N, +)$ *is a group (not necessarily abelian)*

(2) $(N, \cdot)$ *is a semigroup*

(3) $\forall n_1, n_2, n_3 \in N: \ (n_1 + n_2) \cdot n_3 = n_1 \cdot n_3 + n_2 \cdot n_3$

A near-ring $(N, +, *)$ is said to be zero symmetric iff $\forall n \in N : n * 0 = 0$.

Definition 2. *Let* $(N, +, *)$ *be a near-ring and* $(\Gamma, +)$ *be a group.* Γ *is called an N-group iff there exists a multiplication* $\odot$ *such that:*

(1) $\forall \gamma \in \Gamma \forall n_1, n_2 \in N : (n_1 + n_2) \odot \gamma = n_1 \odot \gamma + n_2 \odot \gamma$

(2) $\forall \gamma \in \Gamma \forall n_1, n_2 \in N : (n_1 * n_2) \odot \gamma = n_1 \odot (n_2 \odot \gamma)$.

Preliminaries on N-groups

Let Γ be an N-group and let S be a normal subgroup of Γ. S is called an (N-)ideal of Γ if $\forall n \in N\, \forall \gamma \in \Gamma\, \forall s \in S : n(\gamma + s) - n\gamma \in S$.

N-ideals of the canonical N-group $(N, +)$ are called left ideals of N. A left ideal L of N is an ideal of the near-ring N if $LN \subseteq L$.

Γ is called strongly monogenic if $N\Gamma \neq \{0\}$ and for all $\gamma \in \Gamma$ either $N\gamma = \Gamma$ or $N\gamma = \{0\}$.

Let $\theta_1 =: \{\gamma \in \Gamma \mid N\gamma = \Gamma\}$ and $\theta_0 =: \{\gamma \in \Gamma \mid N\gamma = \{0\}\}$. Then $\Gamma = \theta_0 \cup \theta_1$.

We define $(0 : \Gamma) := \{n \in N \mid \forall \gamma \in \Gamma : n\gamma = 0\}$. Γ is faithful if $(0 : \Gamma) = \{0\}$.

1-primitive near-rings

An N-group Γ is called simple if there do not exist proper N-ideals in Γ.

Definition 3. *A zero symmetric near-ring N is said to be 1-primitive if there exists an N-group Γ which is simple, faithful and strongly monogenic.*

N-groups which are simple and strongly monogenic are called N-groups of type 1.

Definition 4. $\mathbf{J}_1(N) := \cap_{\Gamma \textit{ of type 1}} (0 : \Gamma)$

Note that $\mathbf{J}_1(N) = \{0\}$ if N is 1-primitive.

Strongly monogenic N-groups

Lemma 5. *Let N be a zero symmetric near-ring with a strongly monogenic N-group Γ. Then there exists a greatest proper N-ideal in Γ.*

We denote this greatest proper N-ideal by $\triangle$.

Corollary 6. *Let Γ be a strongly monogenic N-group of a zero symmetric near-ring N. Let $\triangle$ be its greatest proper N-ideal. Then $\Gamma/\triangle$ is an N-group of type 1.*

Theorem 7. *Let N be a zero symmetric near-ring which has a faithful strongly monogenic N-group Γ. Then $\mathbf{J}_1(N)$ is a proper ideal and $N/\mathbf{J}_1(N)$ is a 1-primitive near-ring.*

Lemma 8. *Let $\{0\} \neq L$ be a left ideal of a zero symmetric near-ring N, which is 1-primitive on an N-group Γ. Then Γ is a faithful and strongly monogenic L-group.*

Some lemmas and other by-products

Theorem 9. *Let I be an ideal of a zero symmetric 1-primitive near-ring N. Then I is also a 1-primitive near-ring.*

Lemma 10. *Let N be a zero symmetric near-ring which has a faithful and strongly monogenic N-group Γ. Then, $N\mathbf{J}_1(N) = \{0\}$.*

Lemma 11. *Let N be a zero symmetric near-ring with descending chain condition on N-subgroups of N. Suppose N has a faithful and strongly monogenic N-group Γ and suppose N has a multiplicative right identity. Then $\mathbf{J}_1(N)$ is a proper ideal of N and each proper ideal is contained in $\mathbf{J}_1(N)$.*

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Lemma 12. *Let N be a zero symmetric near-ring with descending chain condition on N-subgroups of N. If there exists an element $e \in N$ with $(0 : e) = \{0\}$, then N has a multiplicative right identity.*

Lemma 13. *Let $\{0\} \neq L$ be a left ideal of a zero symmetric and 1-primitive near-ring N. Suppose L has the descending chain condition on L-subgroups of L. Then L has a multiplicative right identity.*

Theorem 14. *Let $\{0\} \neq L$ be a left ideal of a zero symmetric and 1-primitive near-ring. Suppose L has the descending chain condition on L-subgroups of L. Let I be any proper ideal of L. Then $LI = \{0\}$. Furthermore, there exists a greatest proper ideal Q in L, L/Q is a 1-primitive near-ring and $Q = \mathbf{J}_1(L)$.*

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Corollary 15. *Let N be a finite zero symmetric and 1-primitive near-ring. Let $\{0\} \neq L$ be a left ideal of N. Then, L has a multiplicative right identity and a greatest proper ideal Q. Furthermore, $LQ = \{0\}$.*

Example 16. $R = \{\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \mid a, b, c, d, e, f, g, h, i \in K\}$ *(K a finite field). The ring R acts 1-primitively on the R-group $K_n = \{\begin{pmatrix} j \\ k \\ l \end{pmatrix} \mid j, k, l \in K\}$. $L = \{\begin{pmatrix} a & 0 & 0 \\ d & 0 & 0 \\ g & 0 & 0 \end{pmatrix} \mid a, d, g \in K\}$ is a left ideal of R. $Q = \{\begin{pmatrix} 0 & 0 & 0 \\ d & 0 & 0 \\ g & 0 & 0 \end{pmatrix} \mid d, g \in K\}$ is easily seen to be the greatest ideal of L and $LQ = \{0\}$. $I_1 = \{\begin{pmatrix} 0 & 0 & 0 \\ d & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid d \in K\}$ and $I_2 = \{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ g & 0 & 0 \end{pmatrix} \mid g \in K\}$ are the other ideals of L which both are contained in Q.*