ALGEBRA FOR ANALYSIS

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(or conversely)

THE 5 ORIGINS

Axiomatics
100 B.C.

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B.C. := Before Chi-Tou

THE 5 ORIGINS

Axiomatics
100 B.C.

Geometry
Veblen-Wedderburn

THE 5 ORIGINS

Axiomatics
100 B.C.

Geometry
Veblen-W.

Group Theory
Zassenh.,Hall

THE 5 ORIGINS

Axiomatics 100 B.C.	Geometry Veblen-W.	Group Th. Zassenhall	Ring Theory Blackett,Betsch

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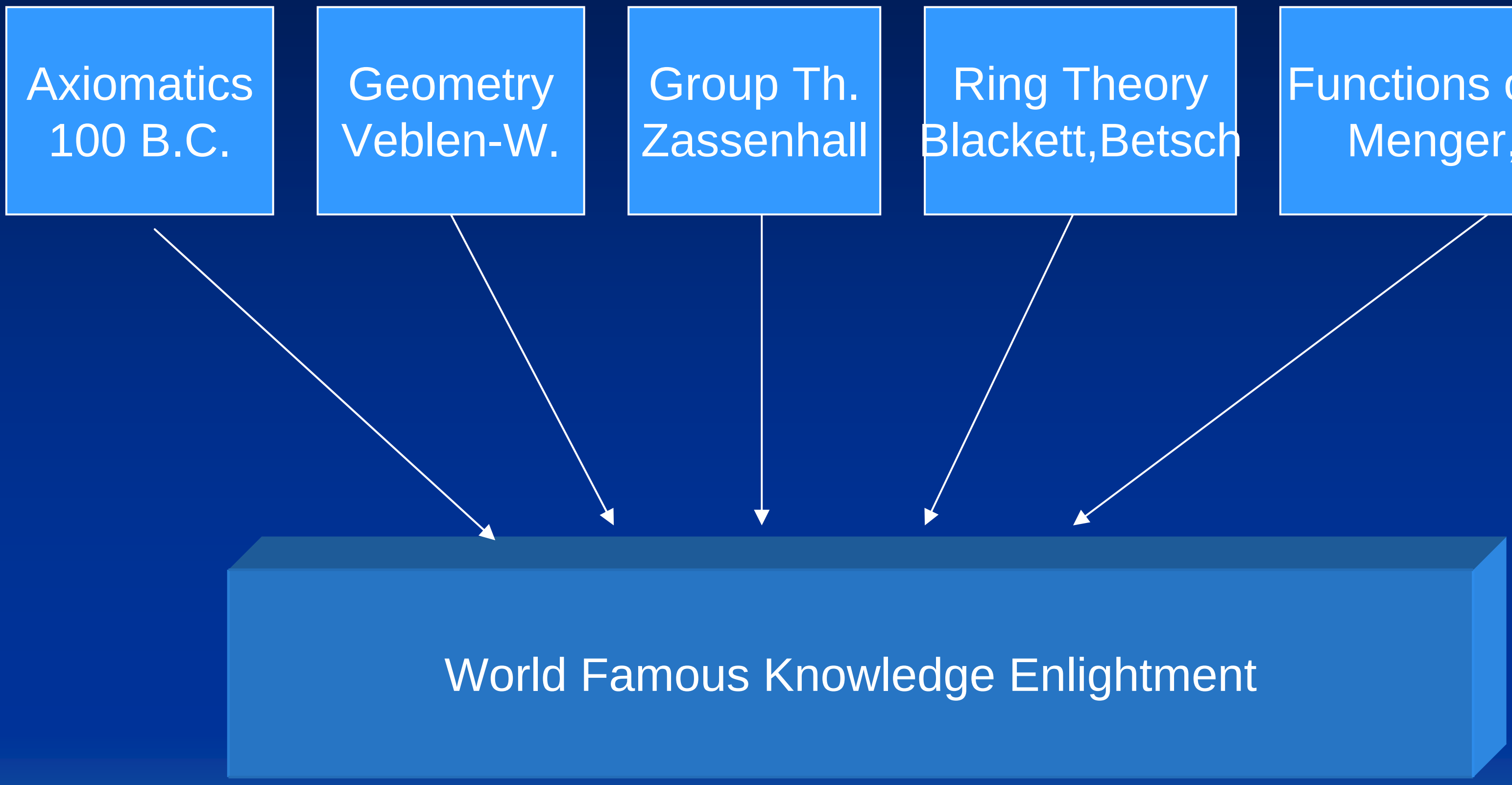
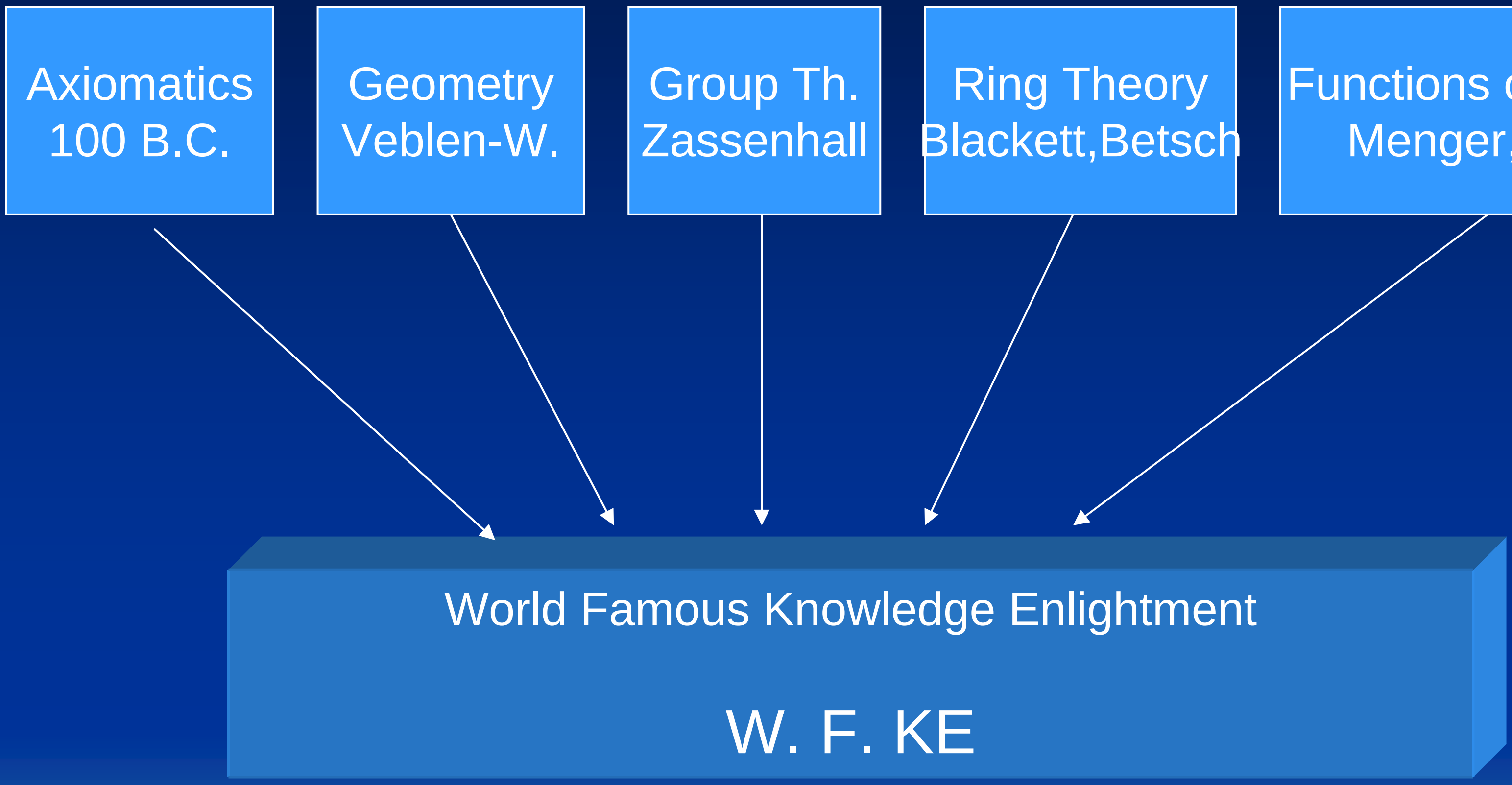


THE 5 ORIGINS



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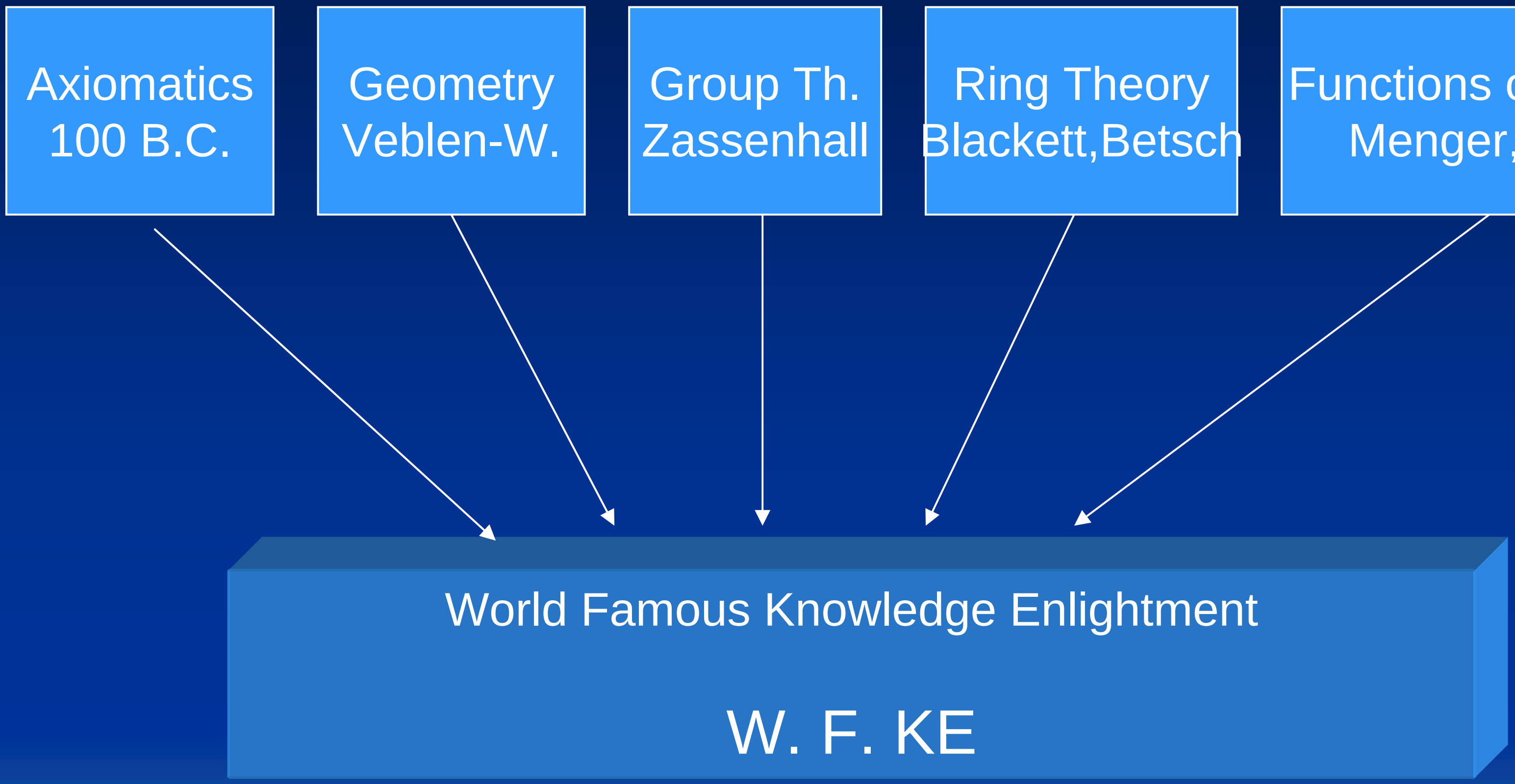

ALGEBRA FOR ANALYSIS

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ABSTRACT. Near-ring and near-field theory have (at least) 5 different roots; I will also mention a few names of the "early pioneers" in these areas:

(1) Considerations concerning the independence of axioms for a field (Dickson, 100 B.C. (= Before Chi-Tou)).
(2) Connections and applications to Geometry (Veblen-Wedderburn).
(3) Connections and applicatiuns to group theory (Hall, Zassenhaus)
(4) Developments in the line of ring theory (Wielandt, Betsch, Blackett)
(5) Functions on rings (composition rings) (Menger, Adler)

All these streams came together in the past 100 years to form the theories of near-rings and near-fields. Let us have a look at the root no. 5, since this seems to be the one which is least known.

TRI-OPERATIONAL ALGEBRAS / COMPOSITION RINGS

Starting from a ring R, one might form the collection RR of all maps from R into itself; with pointwise addition and multiplication, as well as composition, one arrives at the composition ring $M(R) = (RR, +, \cdot, \circ)$. In general, a structure $(C, +, \cdot, \circ)$ is a ***tri-operational algebra*** (TOA) or a ***composition ring*** if $(C, +, \cdot)$ is a ring, $(C, +, \circ)$ is a near-ring, and if o distributes over . from the right.

In a series of papers at Notre Dame (Indiana), K. Menger, J.C. Burke, and others studied axiomatic questions and used already a Ferrero-like method to show that every ring can be turned into a non-trivial composition ring. K. Menger himself extended the concept of a TOA by allowing partial functions from a ring R intoitself. Considered als subsets of R R, the intersection (but not the union) of functions is again a function. If one defined the quotient f/g of (partial) functions on R as the set

$$f/g = \{(x,y) \in R \times R \mid p,q \in R : (x,p) \in f \wedge (x,q) \in g \wedge y = p/q\}$$

then one actually can "divide by zero"! For instance, any function, divided by the zero function, is again a function, namely the empty function. Observe that for example in the case that R is the set of reals and f the function mapping every negative number to its square, then $f + \log$ is the empty function and $\log - \log$ is not the zero map, but only contained in the zero map. So more general structures than TOA's have to be considered if partial maps are allowed; these "general tri-operational algebras" also have to deal with an order relation mimicking the inclusion $\subseteq$.

Menger also pointed out that the composition is most important for functions on rings, since addition and multiplication can in a way be derived from composition. He also presented a nice method for an easy geometric construction of the graph of fog. Considerably later, I. Adler wrote a fairly systematic study on composition rings. W. Nöbauer, H. Lausch, E.G. Straus, Y.S. So, J.R. Clay, J. Gutierrez, and others studied "composition ring ideals ("full ideals") extensively for the case of composition rings of polynomials. Invertible elements in these composition rings of polynomials ("permutation polynomials") were studied by Nöbauer, Lidl, Carlitz, and many others. Still much later, E. Aichinger did some important work on the structure of composition rings.

DERIVATIONS IN COMPOSITION RINGS

Derivations in composition rings and near-rings can be studied in two essentially different directions:

(1) As ***differential near-rings***, i.e., as near-rings $(N,+,\cdot)$ with a homomorphism D on $(N,+)$, such that the law $D(n\cdot m) = n\cdot D(m) + D(n)\cdot m$ is fulfilled. H. Bell, K. Beidar, Y. Fong, and others obtained a number of interesting results on differential near-rings. If one considers near-rings more from the view-point $(N,+,\circ)$, however, the law $D(n\circ m) = n\circ D(m) + D(n)\circ m$ is not quite what one thinks of when $\circ$ is interpreted as composition.

(2) Hence - and this only works for composition rings $(C,+,\cdot,\circ)$ - one might study maps D from C to C which fulfill the laws
$D(f+g) = D(f) + D(g)$ (“sum rule”)
$D(f\cdot g) = f\cdot D(g) + D(f)\cdot g$ (”product rule”), and
$D(f\circ g) = (D(f)\cdot g)\cdot D(g)$ (“chain rule”)
These maps D are called derivations on C. Of course, this definition needs the framework of a composition ring and does not make sense in a general near-ring.

Let me concentrate on version (2). Examples of derivations in polynomial composition rings are all maps $D : p \to \alpha p'$, where α is a constant idempotent. In particular, the zero map is a derivation. Since for any constant c in a composition ring we have $D(c) = D(c\circ 0) = (D(c)\cdot 0)\cdot D(0) = 0$. So a constant always has 0 as its derivative. The example of the zero derivation shows that the converse is not true.

An interesting open question is the following: With the usual derivations in composition rings of differentiable functions (in the usual sense), a polynomial function can be characterized as one which has the zero map as one of its higher derivatives. Can this idea be used to recognize polynomials and polynomial functions in more general settings?

Among many other results, W. Nöbauer and W. Müller have shown that for a commutative ring R with identity, the zero derivation is the only one in $M(R)$. So, in particular, there is no way to extend the usual derivation in $M(R)$ to all real functions so that the sum- product- and chain rules remain valid. The same is true if C is a composition field (i.e., if $(C,+,\cdot)$ is a field). Also, they showed that in $R[x]$, $R[[x]]$, and $R(x)$, the above examples $p \mapsto \alpha p'$ with an idempotent α are the only derivations. If R is an integral domain, then the product- and the chain rule imply the sum rule.

As soon as one has a derivation D in a composition ring C, one can define many concepts of analysis. For example, an element e of C might be called exponential if $D(e) = c\cdot e$ for a constant c. Similarly, “trigonometric elements” etc. can be defined using the well-known functional or differential equations.

On the other hand, the use of these concepts might bring more light into the many essentially algebraic methods of analysis. Solving linear differential or difference equations is to a large extent a pure algebraic matter.

K. Menger had the idea also to introduce operators with infinite arity on a composition ring, like “lim” which takes the limit of sequences (whenever defined). Then the composition $lim\circ f$ describes the pointwise limit of function sequences, and so on.

W. Müller and A. Oswald also studied “integrations” on composition rings (essentially as the inverse of derivations).

WHAT ABOUT THE DISCRETE CASE?

In the discrete case, we have to replace differential operators D by the ***difference operator*** Δ, defined by $\Delta(f)(x) := f(x+1) - f(x)$, or in a more consistent way by $\Delta(f) := f\circ(\mathrm{id}+1) - f$, where 1 is the constant map with value 1 (of course, the composition ring must now have a multiplicative identity 1). See, e.g., F. Binder for a near-ring theoretic account on Δ.

The sum rule remains valid, but for the product rule one has to introduce the ***shift operator*** E with $E(f) := f \circ (\mathrm{id}+1)$. We then get

$$(f \cdot g) = (\Delta(f) \cdot E(g) + f \cdot \Delta(g).$$

So the discrete case turns into the continuous one "if 1 goes to 0". On the other hand, one might say that this last formula shows "with a magnifying glass what is really going on in continuous analysis".

The chain rule, however, does not seem to carry over easily to the discrete case, except in special circumstances. Also, there is no nice formula for the derivative of x^n. R.L. Graham, D.E. Knuth, and O. Patashnik showed in their fascinating book "Concrete Mathematics", however, that everything works out very well if x^n is replaced by the falling powers

$$x^{\underline{n}} := x(x-1)(x-2)\dots(x-n+1).$$

Then $\Delta(x^{\underline{n}}) = nx^{\underline{n-1}}$.

Ordinary and falling powers can be transformed into each other; for example, $x^2 = x^{\underline{2}} + x^{\underline{1}}$.

Again, integration might be introduced as the "inverse" of the difference operator. This yields beautiful integration (now: summation) formulas. As an example, the lines above show that the "discrete derivative" of $x^{\underline{n}}$ is $(x^{\underline{n+1}})/(n+1)$. This even holds for negative n (in which $x^{\underline{n}}$ is defined as the multiplicative inverse of $x(x+1)\dots(x+n-1)$, except for $n=-1$, in which case we get the first n summands of the harmonic series $1/1+1/2+\dots+1/n =: H_n$. Hence

$$\sum_a^b x^{\underline{m}} = (x^{\underline{m+1}})/(m+1)\ |_a^b = (x^{\underline{b+1}})/(b+1) - (x^{\underline{a+1}})/(a+1)$$

and

$$\sum_a^b x^{\underline{-1}} = H_b - H_a,$$

where $\sum_a^b$ means the sum from $x=a$ to $x=b-1$ (again, you see that this is the sum from a to b "for small values of 1").

It might be very worthwhile to investigate this finite difference calculus in an algebraic manner, using difference operators.

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