THE IDEMPOTENT QUIVER OF A NEARRING

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R denotes a 0-symmetric left nearring with 1.

Assume R satisfies dcc on right R-subgroups and $J_2(R)$ is nilpotent.

An idempotent e of R is **primitive** if there does not exist an idempotent $f \in R$ such that $ef = f$ and $fe \neq e$.

A right R-subgroup M of R is **self-monogenic** if $mM = M$ for some $m \in M$.

Theorem 1. Let e be an idempotent of R. TFAE:
1. e is primitive.
2. eR is a minimal self-monogenic right R-subgroup of R.
3. eR is a minimal nonnilpotent right R-subgroup of R.

A set of idempotents $e_1, \ldots, e_n$ of R is **principal** if for any $\overline{r} \in \overline{R} = R/J_2(R)$,

$$\overline{r} = \overline{e_1 r} + \cdots + \overline{e_n r}.$$

A **PPO-set** is a principal set of primitive orthogonal idempotents.

Theorem 2. *Suppose that I is a nilpotent ideal of R and $\varepsilon_1, \ldots, \varepsilon_n$ is a set of primitive orthogonal idempotents of $\overline{R} = R/I$. Then there exists a set of primitive orthogonal idempotents $e_1, \ldots, e_n$ of R such that $\overline{e_i} = \varepsilon_i$.*

Theorem 3. *PPO-sets exist.*

Outline of Proof. Let

$$\overline{R} = R/J_2(R) = A_1 \oplus \cdots \oplus A_n$$

be Wedderburn decomposition of $\overline{R}$ into minimal right ideals. Write

$$\overline{1} = \varepsilon_1 + \cdots + \varepsilon_n,$$

$\varepsilon_i \in A_i$. Now lift $\varepsilon_1, \ldots, \varepsilon_n$ to set of primitive orthogonal idempotents $e_1, \ldots, e_n$ of R. □

Theorem 4. *A set of primitive orthogonal idempotents $e_1, \ldots, e_n$ is a PPO-set $\Leftrightarrow$ $Ann_R(e_1, \ldots, e_n)$ is nilpotent.*

Theorem 5. *A nonnilpotent right R-subgroup of R contains a primitive idempotent.*

Def. Two primitive idempotents e and f are **linked** if there exist primitive idempotents

$$e = e_1, e_2, \ldots, e_n = f$$

such that e_iR and $e_{i+1}R$ have isomorphic R-factors for each i.

Theorem 6. *Let e_1 and e_2 be primitive idempotents of R. e_1R and e_2R have isomorphic R-factors $\Leftrightarrow$ there exists a primitive idempotent g such that $e_1Rg \neq 0$ and $e_2Rg \neq 0$.*

Alt. Def. Two primitive idempotents e and f are linked if there exist primitive idempotents

$$e = e_1, e_2, \ldots, e_n = f$$

such that $e_iRe_{i+1} \neq 0$ or $e_{i+1}Re_i \neq 0$ for each i.

Fix a PPO-set $W = \{e_1, \ldots, e_n\}$ of R.

Let $W_1, \ldots, W_r$ be equivalence classes of W under linkage.

Theorem 7. *R is uniquely expressible as*

$$R = B_1 \oplus \cdots \oplus B_t$$

where each B_i is an indecomposable ideal of R.

The ideals B_i are called the **blocks** of R.

Theorem 8. *If R is tame, $r = t$ and the ideals generated by the equivalence classes W_i are the same as the blocks of R.*

An R-module M is **block indecomposable** if M cannot be written as a direct sum $M_1 \oplus M_2$ where M_1 and M_2 have no isomorphic R-factors.

Theorem 9. *If G is a faithful tame R-module, then G is uniquely expressible as*

$$G = G_1 \oplus \ldots \oplus G_t$$

where each G_i is a block indecomposable R-ideal of G and $G_i = GB_i$.

The R-ideals G_i are called the **blocks** of G.

Theorem 10. *For $e_i, e_j \in W$, TFAE:*
1. $e_i R \simeq e_j R$.
2. $\overline{e_i}\overline{R} \simeq \overline{e_j}\overline{R}$ *where* $\overline{R} = R/J_2(R)$.
3. $e_i r e_j$ *is not in* $J_2(R)$ *for some* $r \in R - J_2(R)$.
4. $e_i R e_j R$ *contains a primitive idempotent.*

Let $e_i \sim e_j$ if one of 1-4 of Theorem 10 holds. Note that if $e_i \sim e_j$, then e_i and e_j are linked.

Choose a set of representatives

$$V = \{e_1, \ldots, e_m\}$$

(relabeling if necessary) of the equivalence classes of W under $\sim$.

The **quiver** of R, denoted $\Gamma(R)$, is the directed graph with vertex set V and directed edges formed by drawing an arrow from e_i to e_j if $e_i R e_j \neq 0$.

Theorem 11. $e_i R e_j \neq 0 \Leftrightarrow e_i J_2(R) e_j \neq 0$.

Theorem 12. *If W' is another PPO-set of R and if V' is a set of equivalence class representatives of W' under $\sim$, then the quivers formed by V and V' are isomorphic.*

Theorem 13. *If R is tame and G is a faithful tame R-module, then the connected components of $\Gamma(R)$ are in one-to-one correspondence with the blocks of R and G.*

Suppose G is a faithful tame R-module.

The **socle** of G, $\mathrm{Soc}(G)$, is the sum of the minimal R-ideals of G.

The **socle series** of G is

$$0 \leq \mathrm{Soc}_1(G) \leq \mathrm{Soc}_2(G) \leq \ldots$$

where $\mathrm{Soc}_1(G) = \mathrm{Soc}(G)$ and $\mathrm{Soc}_{i+1}(G)/\mathrm{Soc}_i(G) = \mathrm{Soc}(G/\mathrm{Soc}_i(G))$.

Theorem 14. *There exists an n such that $\mathrm{Soc}_n(G) = G$ and $\mathrm{Soc}_{i+1}(G)/\mathrm{Soc}_i(G)$ is a direct sum of type 2 R-modules. Also, if e is a primitive idempotent and $\overline{R} = R/J_2(R)$, $\overline{e}\overline{R}$ is isomorphic to a summand of $\mathrm{Soc}_{i+1}(G)/\mathrm{Soc}_i(G)$ for some i.*

Theorem 15. *Suppose $H < K < L$ are R-ideals of G such that K/H and L/K are nonisomorphic type 2 R-modules. Let $e, f \in V$ such that*

$\overline{e}\overline{R} \simeq L/K$ and $\overline{f}\overline{R} \simeq K/H$. If L/H is an indecomposable R-module, then $\Gamma(R)$ contains an arrow from e to f

Theorem 16. *Suppose $e \in V$. If H is a type 2 summand of $\mathrm{Soc}(G)$ such that $\overline{e}\overline{R} \simeq H$ and $G/\mathrm{Soc}(G)$ contains no factor isomorphic to H, then e cannot be an initial vertex of an arrow of $\Gamma(R)$.*

Using SONATA to Calculate Quivers

1. Find the set of nonzero idempotents I of R.
2. Use the definition of primitive idempotent to find the set of primitive idempotents P from I.
3. Filter a PPO-set W from P as follows:
   (i) Choose an element e_1 of P. If $\mathrm{Ann}_R(e_1) \cap P = \emptyset$, done. If not, choose an element e_2 in $\mathrm{Ann}_R(e_1) \cap P$.
   (ii) If $e_2 e_1 = 0$, go to (iii). If not, let
   $$f = (e_1 - e_2 e_1)e_1.$$
   Choose $f_1 \in fR \cap P$.
   Let $e_1 = f_1 f$.
   (iii) Consider $\mathrm{Ann}_R(e_1, e_2) \cap P \ldots$
4. To find V, proceed through $e_1, e_2, \ldots, e_n$, deleting e_j using one of the following approaches:
   (i) if for some $i < j$ we have $e_i R \simeq e_j R$,
   (ii) if for some $i < j$ we have $\overline{e_i}\overline{R} \simeq \overline{e_j}\overline{R}$,
   (iii) if for some $i < j$ we have $e_i r e_j$ is not in $J_2(R)$ for some $r \in R - J_2(R)$
   (iv) if for some $i < j$ we have $e_i R e_j R \cap P \neq \varnothing$.
5. Determine the set of directed edges E of $\Gamma(R)$ by having an arrow from e_i to e_j whenever $e_i R e_j \neq 0$ (or $e_i J_2(R) e_j \neq 0$) and draw the quiver $\Gamma(R)$.