Let R be a subnear-ring of $M_0(\Gamma)$ where Γ is a finite group, and Ω be a minimal faithful R-group.

The **nil-rigid series** of R

$$\{0\} \subseteq L_1 \subset C_1 \subset L_2 \subset \cdots \subset C_{\alpha-1} \subset L_\alpha \subset C_\alpha = R$$

$L_1 := J_0(R)$,

$C_1/L_1 = Soi(R/L_1)$,

$L_2/C_1 = J_0(R/C_1)$,

$\cdots$

$L_\alpha/C_{\alpha-1} = J_0(R/C_{\alpha-1})$

$C_\alpha/L_\alpha = Soi(R/L_\alpha)$.

The integer $\alpha \geq 0$ is called the **nil-rigid length** of R

$$Soi(R) = \bigcap_{K \in \mathcal{K}(\Omega)} (0\colon K),$$

where $\mathcal{K}(\Omega)$ is a class of R-subgroups of Ω with no R-groups of type-0 as direct summands

Motivation

The nil-rigid length -

a measure of how far $J_s(R)$ is from being nilpotent

Thrm. [Hartney]

$J_s(R)$ is nilpotent if and only if $\alpha = 1$

That is,

Nil-rigid length $\alpha > 1 \Rightarrow J_s(R)$ is not nilpotent

There exists a minimal ideal A modulo which $J_s(R)$ is nilpotent, called the s-socle.

ie. $(J_s(R)/A)^m = (0)$

It is known [Hartney] that

$A \subset J_s(R) \cap C_{\alpha-1}$ and $A \not\subset L_{\alpha-1}$

Questions:

How does A relate to the s-socle of $\mathbb{M}_n(R)$?

What if $\alpha < \alpha_{\mathbb{M}_n(R)}$?

When is $\alpha_{\mathbb{M}_n(R)} > \alpha$?

Problem :

No Computer software handles matrix near-rings

Proposition [Betsch]

Let I be an ideal of R , Γ a group and $\nu \in \{0, 1, 2\}$.

(a) If Γ is an R-group with $I \subseteq (0 : \Gamma)$ then $(r + I)\gamma := r\gamma$ makes Γ into an R/I-group.

If ${}_R\Gamma$ is of type-ν, so is ${}_{R/I}\Gamma$.

If ${}_R\Gamma$ is faithful, then so is ${}_{R/I}\Gamma$.

Proposition 1. Let Ω be a faithful R-group. Then $\Gamma \in \mathcal{K}(R/I)$ if and only if $\Gamma \in \mathcal{K}(R)$, with $I \subset (0 : \Gamma)$.

Lemma 1. Consider the following strictly descending chain of R-groups

$$\Delta_1 \supset \Delta_2 \supset \Delta_3 \supset \cdots \supset \Delta_\beta \qquad (1)$$

where $\beta \in \mathbb{N}$. Then each Δ_{i+1} is an (R/I_i)-group, where $I_i = (0 : \Delta_i)$, $i = 1, 2, \ldots, \beta - 1$.

Moreover, R has an ascending chain of ideals,

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \subseteq I_\beta, \qquad (2)$$

corresponding to chain (1).

Defn.1 Let Ω be a faithful finite R-group, and let

$$\Omega_1 \supset \Omega_2 \supset \dots \supset \Omega_{\beta-1} \supset \Omega_\beta, \qquad (3)$$

be a strictly descending chain of R-subgroups of Ω such that

Ω_1 is a maximal R-group of type-0, and

Ω_β is an R-group of type-s.

Chain (3) is an **Ω-chain** if for every pair (Ω_i, Ω_{i+1}), $1 \le i < \beta$, of consecutive members of chain (3) :

whenever Ω_i is of type-0 then $\Omega_{i+1} \in \mathcal{K}(\Omega)$ and it is a maximal R-group in $\mathcal{K}(\Omega)$ contained in Ω_i; or

whenever $\Omega_i \in \mathcal{K}(\Omega)$ then Ω_{i+1} is a maximal type-0 R-group contained in Ω_i.

The **Ω-length** of an Ω-chain (3) is $\frac{\beta-1}{2}$.

A $\mathcal{KG}$-**chain** of Ω is an Ω-chain of maximal Ω-length.

The Ω-length of a $\mathcal{KG}$-chain is called the $\mathcal{KG}$-**length** of Ω.

Defn.2 Let Ω be a finite faithful R-group and let

$$\Omega_{1,1} \supset \Omega_{1,2} \supset \ldots \supset \Omega_{1,(l(1)-1)} \supset \Omega_{1,l(1)},$$

$$\Omega_{2,1} \supset \Omega_{2,2} \supset \ldots \supset \Omega_{2,(l(2)-1)} \supset \Omega_{2,l(2)},$$

$$\ldots$$

$$\ldots$$

$$\Omega_{\lambda,1} \supset \Omega_{\lambda,2} \supset \ldots \supset \Omega_{\lambda,(l(\lambda)-1)} \supset \Omega_{\lambda,l(\lambda)},$$

be a list of all Ω-chains.

Define the j-**th** Ω-**ideal**, T_j, of R to be the intersection of annihilators of the j-th members of all Ω-chains of lengths $l(\rho) \geq j$.

That is,

$$T_j := \bigcap_{j \leq l(\rho)} (\, 0 : \Omega_{\rho,j}\,), \quad j = 1,2,\ldots,\tau(\Omega),$$

where $\tau(\Omega) = \max\{\, l(i) \mid i = 1,2,..,\lambda\,\}$.

We also define $T_0 := (0)$ and $T_{\tau(\Omega)+1} := R$.

Thrm.1 Let Ω be a finite faithful R-group.
The Ω-ideals form an ascending series of ideals of R. That is,

$$T_1 \subset T_2 \subset T_3 \subset \dots \subset T_{\tau(\Omega)},$$

where T_j is as in Definition 2.

Thrm.2 Let $\Omega := (R,+)$, and T_1 be the 1^{st} Ω-ideal of R. Then,

$$J_0(R) = T_1 = L_1,$$

where L_1 is as in the definition of the nil-rigid series.

Thrm.3 Let $\Omega := (R,+)$, and T_2 be the 2^{nd} Ω-ideal of R. Let L_1 and C_1 be as in the definition of the nil-rigid series of R. Then

$$T_2/L_1 \supseteq Soi\,(R/L_1)\,.$$

Moreover $T_2 \supseteq C_1$.

Thrm.4 Let $\Omega := (R, +)$, and ideals C_1 and L_2 be as in the definition of the nil-rigid series of R. Let T_3 be the 3rd Ω-ideal of R. If $\tau(\Omega) \geq 3$ then

$$T_3/C_1 \supseteq J_0\,(R/C_1)\,.$$

Moreover $T_3 \supset L_2$.

Lemma.2 Let the nil-rigid length $\alpha > 3$, and $\Omega := (R, +)$ be a faithful R-group with $\mathcal{KG}$-length $\frac{\tau(\Omega)-1}{2} > 2$. For any $\mathcal{KG}$-chain

$$\Omega_{i,\,1} \supset \Omega_{i,\,2} \supset \ldots \supset \Omega_{i,\,(\tau(\Omega)-1)} \supset \Omega_{i,\,\tau(\Omega)},$$

(a) If j is even then $\Omega_{i,\,j}$ is in $\mathcal{K}(R/L_{j/2})$; and
(b) If j is odd then $\Omega_{i,\,j}$ is a type-0 $(R/C_{(j-1)/2})$-group.

Thrm.A. Let the nil-rigid length $\alpha > 3$, and $\Omega := (R, +)$ be a faithful R-group. Let T_j be the j-th Ω-ideal of R, for $2 \leq j \leq \tau(\Omega) + 1$, and $L_{(j/2)}$ and $C_{(j-1)/2}$ be ideals of R in the nil-rigid series.

(a) If j is even then $T_j/L_{(j/2)} \supset Soi\left(R/L_{(j/2)}\right)$; and

(b) If j is odd then $T_j/L_{(j-1)/2} \supset J_0\left(R/C_{(j-1)/2}\right)$.

Thrm.B. Let $\Omega := (R, +)$. The series of Ω-ideals,

$$T_1 \subset T_2 \subset T_3 \subset \ldots \subset T_{\tau(\Omega)} \subset T_{\tau(\Omega)+1} = R,$$

and the nil-rigid series of R,

$$L_1 \subset C_1 \subset L_2 \subset C_2 \subset \ldots \subset L_\alpha \subset C_\alpha = R,$$

are related as

$$T_{2i-1} \supseteq L_i \quad \text{and} \quad T_{2i} \supseteq C_i,$$

where $1 < i < \alpha$.

Thrm.5 Let $\Omega := (R,+)$ be a faithful R-group with $\mathcal{KG}$-length $k = \frac{\tau(\Omega)-1}{2}$. For any two consecutive members of a $\mathcal{KG}$-chain, $\Omega_{i,j}$ and $\Omega_{i,j+1}$,

if $\Omega_{i,j+1}$ is an R-kernel to $\Omega_{i,j} \in \mathcal{K}(\Omega)$,

then

j is even and $\Omega_{i,j}/\Omega_{i,j+1}$ is an $(R/C_{j/2})$-group.

Thrm.6 Let $\Omega := (R,+)$, be a faithful R-group with the Ω-ideal series and the nil-rigid series as in Theorem B.

If Ω has a $\mathcal{KG}$-length $k = \frac{\tau(\Omega)-1}{2}$, then the nil-rigid length of R,

$$\alpha \geq k+1.$$

Conjecture:

If the $\mathcal{KG}$-length of Ω is k, then the nil-rigid length

$$\alpha = k+1.$$