Is every Neardomain a Nearfield?

Hubert Kiechle

Definition: $(F,+,\cdot)$ is called a neardomain if

- $(F,+)$ is a loop with two-sided inverses (i.e., $a+b=0 \implies b+a=0$)
- $(F^*,\cdot)$ is a group
- $0a=0$
- $a(b+c)=ab+ac$
- $\exists d_{a,b} \in F$ such that $a+(b+c)=(a+b)+d_{a,b}\cdot c$

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Example: Every nearfield is a neardomain.

Then the group

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$$F \mapsto T_2(F) \quad \text{is categorial}$$

More precisely, T_2 is an equivalence of categories.

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Sharply k-transitive groups besides S_k, S_{k+1}, A_{k+2}

$k = 4$: M_{11}

$k = 5$: M_{12}

$k \geq 6$: none

Characterization

$$
\begin{aligned}
& F \text{ is a nearfield} \\
\Longleftrightarrow\ & T_2(F) \text{ contains an abelian, normal subgroup } A \neq 1 \\
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& \text{acting fixed-point-free on } F
\end{aligned}
$$

In this case A is isomorphic to $(F, +)$

Planarity

A neardomain F is called planar if the equation

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$$\begin{aligned} & F \text{ is a planar neardomain} \\ \iff & F \text{ is a planar nearfield} \\ \iff & \text{the fixed-point-free elements of } T_2(F) \text{ form a subgroup} \end{aligned}$$

Additive Structure

$(F, +)$ is a K-loop (or Bruck loop), i.e.,

- There exists 0 in F;
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- $-(a + b) = (-a) + (-b)$.

Properties

Let $a, b \in F$. Then $\delta_{a,b} : F \to F$, defined by

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Theorem: $\delta_{a,b} \in \mathrm{Aut}\, F$.
(FUNK, NAGY '93; GOODAIRE, ROBINSON '94; KREUZER '98)

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$\mathrm{char}\, F$ is either a prime or 0

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- If F is finite, then F is a nearfield (Zassenhaus '34)
- If $T_2(F)$ is locally compact, connected, then F is a nearfield (Tits '52)
- If $\operatorname{char} F = 3$, then F is a nearfield (Kerby, Wefelscheid '72)
- If F is a KT-field with $\operatorname{char} F \equiv 1 \mod 3$, then F is a nearfield (Kerby '74)

Generalization

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(G, P) has many involutions if

- $J := \{g \in G; g^2 = 1\}$ is transitive
- $|\Omega \cap J^{\#}| \leq 1$

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G can be reconstructed from L.

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G is sharply 2-transitive, iff in addition H acts transitive.

Examples

- $G = \mathrm{PSL}(2, R)$ acting on the set of positive definite, symmetric matrices of determinant 1 by $X \mapsto AXA^{\mathrm{T}}$ is a Frobenius group with many involutions.

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$$a \oplus b := \frac{a + b}{1 + ta\bar{b}}$$

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  - If $\operatorname{char} K = 2$, then L is of exponent 2.
  - If $\operatorname{char} K \neq 2$, then every element of L has infinite order.