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ELEMENTARY EQUIVALENCE OF ENDOMORPHISM RINGS OF ABELIAN p-GROUPS

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In our talk we will consider elementary properties (i.e. properties which are definable in the first order language) of endomorphism rings of abelian p-groups.

The first result on relationship of elementary properties of some models with elementary properties of derivative models was proved by A. I. Maltsev in 1961. He proves that the groups $G_n(K)$ and $G_m(L)$ ($G =$ GL, SL, PGL, PSL, $n,m \geq 3$, K and L are fields of characteristic 0) are elementarily equivalent if and only if $m = n$ and the fields K and L are elementarily equivalent.

This theory was continued in 1992 when with the help of the construction of ultraproduct and the isomorphism theorem of S. Shelah Kostya Beidar and Alexandr V. Mikhalev formulate a general approach to problems of elementary equivalence of different algebraic structures, and generalize Maltsev theorem for the case when K and L are skewfields and associative rings.

In 1998–2004 Elena Bunina continued to study some problems of this type. She generalized the results of Maltsev for unitary linear groups over skewfields and associative rings with involutions, and also for Chevalley groups over fields.

In 2000 Vladimir Tolstykh studied relationship between second order properties of skewfields and first order properties of automorphism groups of infinite dimensional linear spaces over them. In 2003 Elena Bunina and Alexandr V. Mikhalev studied relationship between second order properties of associative rings and first order properties of categories of modules, endomorphism rings, automorphism groups, projective spaces of infinite rank over these rings.

Now we study relationship between second order properties of abelian p-groups and first order properties of their endomorphism rings.

We will describe the second order group language $\mathcal{L}_2$, and also its restriction $\mathcal{L}_2^{\varkappa}$ by some cardinal number $\varkappa$, and then introduce the *expressible rank* r_{exp} of the abelian group A, represented as the direct sum $D \oplus G$ of its divisible and reduced components, as the maximum of powers of the group D and some basic subgroup B of A, i. e. $r_{exp} = \max(r(D), r(B))$. Then we can formulate the main theorem:

Theorem 1. *If A_1 and A_2 are abelian p-groups, $\varkappa_1 = r_{exp}(A_1)$, $\varkappa_2 = r_{exp}(A_2)$, then elementary equivalence of endomorphism rings $End(A_1)$ and $End(A_2)$ implies*

$$Th_2^{\varkappa_1}(A_1) = Th_2^{\varkappa_2}(A_2).$$

Note that $r_{exp}(A) = |A|$ in all cases except the case when $|D| < |G|$, any basic subgroup of A is countable, and the group G itself is uncountable. In this case $r_{exp}(A) = \omega$.

Then we prove two "inverse implications" of the main theorem:

Theorem 2. *For any abelian groups A_1 and A_2 if the groups A_1 and A_2 are equivalent in the second order language $\mathcal{L}_2$ then the rings $End(A_1)$ and $End(A_2)$ are elementarily equivalent.*

Theorem 3. *If abelian groups A_1 and A_2 are reduced and their basic subgroups are countable then $Th_2^{\omega}(A_1) = Th_2^{\omega}(A_2)$ implies $End(A_1) \equiv End(A_2)$.*

Therefore for all abelian groups, except the case $A = D \oplus G$, $D \neq 0$, $|D| < |G|$, and $|G| > \omega$, a basic subgroup in A is countable, elementary equivalence of the rings $End(A_1)$ and $End(A_2)$ is equivalent to

$$Th_2^{\varkappa_1}(A_1) = Th_2^{\varkappa_2}(A_2).$$