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CLASSIFICATION OF TOTAL VALUATION RINGS OF $K(X;\sigma,\delta)$ CONTAINING X, X^{-1}

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Let σ be an endomorphism of a skew field K. A (left) σ-derivation of K is any additive map $\delta : K \to K$ such that $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all $a,b \in K$. Then there exists a ring S, containing K as a subring, such that S is a free left K-module with a basis of the form 1, X, X^2, $\ldots$, and $Xa = \sigma(a)X + \delta(a)$ for all $a \in K$. The ring S is denoted $K[X;\sigma,\delta]$ and is called a skew polynomial ring of K. It is known that the ring $K[X;\sigma,\delta]$ is a principal left ideal domain, so that $K[X;\sigma,\delta]$ is a left Ore domain. We denote $K(X;\sigma,\delta)$ as the quotient division ring of $K[X;\sigma,\delta]$. We say that the pair (K,V) is a *valued skew field* if K is a skew field with the subring V such that $a \in K\backslash V$ implies $a^{-1} \in V$, i.e., V is a *total valuation* ring of K. We consider the extensions of V in $K(X;\sigma,\delta)$, i.e., the total valuation ring R of $K(X;\sigma,\delta)$ with $R \cap K = V$. Let R be an extension of V in $K(X;\sigma,\delta)$ and $J(V)$ the Jacobson radical of V and $J(R)$ be the Jacobson radical of R. Then since $J(V) = J(R) \cap K$, $V/J(V)$ is a subring of $R/J(R)$. If $\pi_V : V \to V/J(V)$ is the canonical map, one put $\pi_V(a) = \overline{a}$ for all $a \in V$, and also $\pi_R : R \to R/J(R)$. An element $\overline{f}$ in $R/J(R)$ is called (left) *transcendental* over $V/J(V)$ if for any natural number n, and any elements $\overline{a_0}, \overline{a_1}, \cdots, \overline{a_n} \in V/J(V)$, $\overline{a_0} + \overline{a_1}\,\overline{f} + \cdots + \overline{a_n}\,\overline{f}^n = \overline{0}$ implies $\overline{a_i} = \overline{0}$ for all i ($i = 0, \ldots, n$). (σ,δ) is called *compatible* with V if $\sigma(V) \subseteq V$, $\sigma(J(V)) \subseteq J(V)$, and $\delta(V) \subseteq V$, $\delta(J(V)) \subseteq J(V)$ in order to characterize the existence of an extension of V in which $\overline{X}$ is transcendental over $V/J(V)$. If V is a total valuation ring and (σ,δ) is compatible with V, then $J(V)[X;\sigma,\delta]$ is localizable and $R^{(1)} = V[X;\sigma,\delta]_{J(V)[X;\sigma,\delta]}$ is a total valuation ring with $R^{(1)} \cap K = V$ and $\overline{X}$ is transcendental over $V/J(V)$.

We shall show that there exists a total valuation ring R of $K(X;\sigma,\delta)$ which is an extension of V and $\overline{X}$ is transcendental over $V/J(V)$ if and only if (σ,δ) is compatible with V. R is exactly equal to $R^{(1)}$. If X and X^{-1} are contained in R and $\overline{X}$ is algebraic over $V/J(V)$, the structure of R also can be clarified, so we can classify the total valuation ring R of $K(X,\sigma,\delta)$ which containing X and X^{-1}.