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AFFINE GEOMETRY AND ENGEL-LIKE IDENTITIES CHARACTERIZING FINITE SOLVABLE GROUPS

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We characterize solvable groups in the class of finite groups by identities in two variables.

We define a sequence: $u_1(x,y) := x^{-2}y^{-1}x$, and inductively

$$u_{n+1}(x,y) := [x u_n(x,y) x^{-1}, y u_n(x,y) y^{-1}].$$

Our main result is

Theorem 1. *A finite group G is solvable if and only if for some n the identity $u_n(x,y) = 1$ holds in G.*

Although Theorem 1 is a purely Group-Theoretic result, its proof involves surprisingly diverse methods of Ring Theory, Algebraic Geometry, Group Theory, and Computer Algebra.

The "only if" part of the Theorem is trivial. The non-trivial direction of the Theorem follows immediately from the following

Theorem 2. *Let G be a finite non-abelian simple group. Then there are elements x and y of G such that $u_1(x,y) \neq 1$ and $u_1(x,y) = u_2(x,y)$.*

Using Thompson's list of the minimal simple non-solvable groups we only need to prove Theorem 2 for the groups G in the following list.

(1) $G = \mathrm{PSL}(3, \mathbb{F}_3)$,
(2) $G = \mathrm{PSL}(2, \mathbb{F}_q)$ where $q \geq 4$ ($q = p^n$, p a prime),
(3) $G = \mathrm{Sz}(2^n)$, $n \geq 3$ and odd.

Here $\mathbb{F}_q$ stands for the finite field with q elements and $\mathrm{Sz}(2^n)$ ($n \geq 3$) denote the Suzuki groups.

For small groups from this list it is a computer task to verify Theorem 2. There are for example altogether 44928 suitable pairs x and y in the group $\mathrm{PSL}(3, \mathbb{F}_3)$.

The general idea of our proof can be roughly described as follows. For a group G in the above list, using a matrix representation over $\mathbb{F}_q$ we interpret solutions of the equation $u_1(x,y) = u_2(x,y)$ as $\mathbb{F}_q$-rational points of an affine variety V_G.

In case (2) this variety does not depend on n. By Computational Ring Theory methods we prove that it is an absolutely irreducible curve and then we use Hasse-Weil type estimates for the number of rational points on a variety defined over a finite field, which guarantee the existence of such points for big q.

In case (3) variety V_G for Suzuki group G depends on n. We manage to find a closed affine absolutely irreducible affine surface V of $\mathbb{A}^8$ together with the endomorphism α of $\mathbb{A}^8$ such that

(1) V is invariant under α,
(2) α^2 is the Frobenius map, and
(3) the elements of the group $G = \mathrm{Sz}(2^n)$ are precisely the points of V which are fixed for α^n.

In order to use the Lefschetz trace formula and to show that such points exist, we had to analyze this affine set, i.e. to find its singular locus, to estimate betti numbers and to understand what happens at infinity.