DERIVATIONS AND SKEW POLYNOMIAL RINGS

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Let \( R \) be a prime ring and \( d \) a derivation of \( R \). Let \( E(R,+) \) denote the ring of additive endomorphisms of \( R \) endowed with pointwise addition and composition multiplication. Obviously, \( d \in E(R,+) \).

For \( a \in R \), let \( a_L \in E(R,+) \) denote the left multiplication \( a_L : x \in R \mapsto ax \in R \). Let \( S \) be the subring of \( E(R,+) \) generated by \( d \) and all \( a_L, a \in R \). We shall compute the prime radical and minimal prime ideals of \( S \) as follows: Let \( R[x;d] \) be the skew polynomial ring with the multiplication rule: \( xr = rx + d(r) \) for \( r \in R \). Since \( da_L = d(a)_L + a_Ld \) for \( a \in R \), the map

\[
\varphi : a_0x^n + \cdots + a_{n-1}x + a_n \mapsto (a_0)_Ld^n + \cdots + (a_{n-1})_Ld + (a_n)_L \in S
\]

is a surjective ring homomorphism. \( \varphi \). Then \( R[x;d]/\mathcal{A} \cong S \), where \( \mathcal{A} \) is the kernel of \( \varphi \). Let \( \mathcal{P} \) be the ideal of \( R[x;d] \) including \( \mathcal{A} \) such that \( \mathcal{P}/\mathcal{A} \) is the prime radical of \( R[x;d]/\mathcal{A} \). Our aim is to describe in the ring \( R[x;d] \) the ideals \( \mathcal{A}, \mathcal{P} \) and also the minimal prime ideals over \( \mathcal{A} \) in the following way: Let \( Q \) denote the symmetric Martindale quotient ring of \( R \). The center \( C \) of \( Q \) is called the extended centroid of \( R \). Let \( C^{(d)} := \{ \alpha \in C : d(\alpha) = 0 \} \). Matczuk has shown that the center of \( Q[x;d] \) assumes the form \( C^{(d)}[\zeta] \). Given \( f(\zeta) \in C[\zeta] \), we let

\[
\langle f(\zeta) \rangle := R[x;d] \cap f(\zeta)Q[x;d].
\]
We show that $\mathcal{A}$, $\mathcal{P}$ and all minimal prime ideals over $\mathcal{A}$ are of the above form and we compute these center elements $f(\zeta)$ explicitly.

Here is an application: Assume that $d$ is a nilpotent derivation. Let $m$ be the least integer such that $d^m(R)c = 0$ for some $0 \neq c \in R$. Let $(x^m)$ denote the ideal of $R[x; d]$ generated by $x^m$. Then $(x^m) \cap R = 0$. We extend $(x^m)$ to an ideal $\mathcal{M}$ of $R[x; d]$ maximal with respect to the property $\mathcal{M} \cap R = 0$. The quotient ring $R[x; d]/\mathcal{M}$ is an extension of $R$ and is called the $d$-extension of $R$. Using the above results, we show that $\mathcal{M} = \mathcal{P}$. So the $d$-extension of $R$ exists uniquely and is isomorphic canonically to the quotient ring of $S$ modulo its prime radical.