Right Gaussian rings relative to a monoid

Michał Ziembowski

Warsaw University of Technology

Taipei
July 12, 2011

Based on a joint work with R. Mazurek

- Let R be a commutative ring, and denote by $Q(R)$, the total ring of quotients of R. An ideal I of R, is *invertible* if $I \cdot I^{-1} = R$, where $I^{-1} = \{r \in Q(R) : rI \subseteq R\}$.

- Let R be a commutative ring, and denote by $Q(R)$, the total ring of quotients of R. An ideal I of R, is *invertible* if $I \cdot I^{-1} = R$, where $I^{-1} = \{r \in Q(R) : rI \subseteq R\}$.
- Recall that a commutative ring R is a Prüfer ring (if R is domain, then R is called Prüfer domain) if every finitely generated regular ideal of R is invertible.

- Let R be a commutative ring, and denote by $Q(R)$, the total ring of quotients of R. An ideal I of R, is *invertible* if $I \cdot I^{-1} = R$, where $I^{-1} = \{r \in Q(R) : rI \subseteq R\}$.
- Recall that a commutative ring R is a Prüfer ring (if R is domain, then R is called Prüfer domain) if every finitely generated regular ideal of R is invertible.
- For a commutative domain R we have

$$R \text{ is semihereditary } \Leftrightarrow$$

$$\Leftrightarrow R \text{ has weak dimension less or equal to one } \Leftrightarrow$$

$$\Leftrightarrow R \text{ is distributive } \Leftrightarrow R \text{ is Gaussian } \Leftrightarrow R \text{ is Prüfer}$$

- For a commutative ring R we have

$$R \text{ is semihereditary } \Rightarrow$$

$$\Rightarrow R \text{ has weak dimension less or equal to one } \Rightarrow$$

$$\Rightarrow R \text{ is distributive } \Rightarrow R \text{ is Gaussian } \Rightarrow R \text{ is Prüfer}$$

For a ring R and a polynomial $f \in R[x]$, let $c_r(f)$ denote the right ideal of R generated by the coefficients of f. Obviously, for any $f, g \in R[x]$ we have $c_r(fg) \subseteq c_r(f)c_r(g)$.

Definition 1

A ring R is *right Gaussian* if $c_r(fg) = c_r(f)c_r(g)$ for any $f, g \in R[x]$.

Recall that a ring R is an *Armendariz ring* if whenever the product of two polynomials over R is zero, then the products of their coefficients are all zero, that is, for any $f = \sum_{i=0}^{m} a_i x^i,\ g = \sum_{j=0}^{n} b_j x^j \in R[x]$, if $fg = 0$, then $a_i b_j = 0$ for all i, j.

Recall that a ring R is an *Armendariz ring* if whenever the product of two polynomials over R is zero, then the products of their coefficients are all zero, that is, for any $f = \sum_{i=0}^{m} a_i x^i, \ g = \sum_{j=0}^{n} b_j x^j \in R[x]$, if $fg = 0$, then $a_i b_j = 0$ for all i, j.

Facts about right Gaussian rings

Recall that a ring R is an *Armendariz ring* if whenever the product of two polynomials over R is zero, then the products of their coefficients are all zero, that is, for any $f = \sum_{i=0}^{m} a_i x^i,\ g = \sum_{j=0}^{n} b_j x^j \in R[x]$, if $fg = 0$, then $a_i b_j = 0$ for all i, j.

Facts about right Gaussian rings

- *If a ring R is right Gaussian, then R is right duo.*

Recall that a ring R is an *Armendariz ring* if whenever the product of two polynomials over R is zero, then the products of their coefficients are all zero, that is, for any $f = \sum_{i=0}^{m} a_i x^i,\ g = \sum_{j=0}^{n} b_j x^j \in R[x]$, if $fg = 0$, then $a_i b_j = 0$ for all i, j.

Facts about right Gaussian rings

- *If a ring R is right Gaussian, then R is right duo.*
- *If a ring R is right Gaussian, then so is any homomorphic image of R*

Recall that a ring R is an *Armendariz ring* if whenever the product of two polynomials over R is zero, then the products of their coefficients are all zero, that is, for any $f = \sum_{i=0}^{m} a_i x^i,\ g = \sum_{j=0}^{n} b_j x^j \in R[x]$, if $fg = 0$, then $a_i b_j = 0$ for all i, j.

Facts about right Gaussian rings

- *If a ring R is right Gaussian, then R is right duo.*
- *If a ring R is right Gaussian, then so is any homomorphic image of R*
- *A direct product ring $\prod_{i\in I} R_i$ is right Gaussian if and only if each component ring R_i is.*

Recall that a ring R is an *Armendariz ring* if whenever the product of two polynomials over R is zero, then the products of their coefficients are all zero, that is, for any $f = \sum_{i=0}^{m} a_i x^i,\ g = \sum_{j=0}^{n} b_j x^j \in R[x]$, if $fg = 0$, then $a_i b_j = 0$ for all i, j.

Facts about right Gaussian rings

- *If a ring R is right Gaussian, then R is right duo.*
- *If a ring R is right Gaussian, then so is any homomorphic image of R*
- *A direct product ring $\prod_{i\in I} R_i$ is right Gaussian if and only if each component ring R_i is.*
- *A ring R is right Gaussian if and only if R is right duo and every homomorphic image of R is Armendariz.*

The main tool for commutative rings:

The main tool for commutative rings:

A ring R is Gaussian if and only if R_M is Gaussian for each maximal ideal M of R

The main tool for commutative rings:

A ring R is Gaussian if and only if R_M is Gaussian for each maximal ideal M of R

What about noncommutative case?

The main tool for commutative rings:

A ring R is Gaussian if and only if R_M is Gaussian for each maximal ideal M of R

What about noncommutative case?

Theorem 2 (R.Mazurek, M.Z. (2011))

Let R be a right Gaussian ring, P an ideal of R such that $S = R \setminus P$ is a right denominator set in R, and R_S a right ring of quotients with respect to S. Then the following conditions are equivalent:

(1) *R_S is right Gaussian.*

(2) *R_S is right duo.*

(3) *For any $a \in R$ we have $Sa \subseteq aS$ or $as = 0$ for some $s \in S$.*

- For a commutative ring R we have

$$R \text{ is semihereditary } \Rightarrow$$

$$\Rightarrow R \text{ has weak dimension less or equal to one } \Rightarrow$$

$$\Rightarrow R \text{ is distributive } \Rightarrow R \text{ is Gaussian } \Rightarrow R \text{ is Prüfer}$$

- For a commutative ring R we have

$$R \text{ is semihereditary } \Rightarrow$$

$$\Rightarrow R \text{ has weak dimension less or equal to one } \Rightarrow$$

$$\Rightarrow R \text{ is distributive } \Rightarrow R \text{ is Gaussian } \Rightarrow R \text{ is Prüfer}$$

- Is it true that if R is right distributive ring, then R is right Gaussian?

- For a commutative ring R we have

$$R \text{ is semihereditary } \Rightarrow$$

$$\Rightarrow R \text{ has weak dimension less or equal to one } \Rightarrow$$

$$\Rightarrow R \text{ is distributive } \Rightarrow R \text{ is Gaussian } \Rightarrow R \text{ is Prüfer}$$

- Is it true that if R is right distributive ring, then R is right Gaussian?

Answer: NO!

Theorem 3 (R.Mazurek, M.Z. (2011))

If R is a right duo right distributive ring, then R is right Gaussian.

Definition 4 (Zhongkui Liu (2005))

Let M be a monoid. A ring R is called an M-Armendariz ring, if whenever elements $a = a_1 s_1 + a_2 s_2 + ... + a_n s_n;\ b = b_1 t_1 + b_2 t_2 + ... + b_m t_m \in R[M]$ satisfy $ab = 0$, then $a_i b_j = 0$ for each $i; j$.

Definition 4 (Zhongkui Liu (2005))

Let M be a monoid. A ring R is called an M-Armendariz ring, if whenever elements
$a = a_1 s_1 + a_2 s_2 + ... + a_n s_n;\ b = b_1 t_1 + b_2 t_2 + ... + b_m t_m \in R[M]$
satisfy $ab = 0$, then $a_i b_j = 0$ for each $i; j$.

Recall that a monoid M is called a *unique product monoid* (or a u.p.-monoid) if for any two nonempty finite subsets $X; Y \subseteq M$ there exist $x \in X$ and $y \in Y$ such that $xy \neq x'y'$ for every $(x', y') \in X \times Y \backslash (x, y)$; the element xy is called a *u.p.-element* of $XY = \{st : s \in X, t \in Y\}$.

Definition 4 (Zhongkui Liu (2005))

Let M be a monoid. A ring R is called an M-Armendariz ring, if whenever elements $a = a_1 s_1 + a_2 s_2 + \ldots + a_n s_n; b = b_1 t_1 + b_2 t_2 + \ldots + b_m t_m \in R[M]$ satisfy $ab = 0$, then $a_i b_j = 0$ for each $i; j$.

Recall that a monoid M is called a *unique product monoid* (or a u.p.-monoid) if for any two nonempty finite subsets $X; Y \subseteq M$ there exist $x \in X$ and $y \in Y$ such that $xy \neq x'y'$ for every $(x', y') \in X \times Y \backslash (x, y)$; the element xy is called a *u.p.-element* of $XY = \{st : s \in X, t \in Y\}$.

Theorem 5 (M.Z. (2011))

If R is a right duo right distributive ring and M is a u.p.-monoid, then R is M-Armendariz.

Question

Does there exist an Armendariz ring R such that for some u.p.-monoid M, R is not M-Armendariz.

Theorem 6 (D.D. Anderson, V. Camillo, (1998))

For a commutative ring R, the following conditions are equivaent:

(1) $R[[x]]$ *is Gaussian*

(2) $R[[x]]$ *is distributive*

(3) $R[[x]]$ *has weak dimension less or equal to one*

(4) R *is* $\aleph_0$*-injective von Neumann regular*

Theorem 7 (R. Mazurek, M.Z., (2011))

Let σ be an endomorphism of a ring R. Then the following conditions are equivalent:

(1) $R[[x;\sigma]]$ *right Gaussian.*

(2) $R[[x;\sigma]]$ *is right distributive and right duo.*

(3) $R[[x;\sigma]]$ *has weak dimension less or equal to one and* $R[[x;\sigma]]$ *is right duo.*

(4) R *is* $\aleph_0$*-injective strongly regular,* σ *is bijective and* $\sigma(e) = e$ *for any* $e = e^2 \in R$.

THANK YOU FOR YOUR ATTENTION.