Some recent work on clean rings:

1. The structure of a class of clean rings
2. An application of a theorem on clean endomorphism rings

by

Yiqiang Zhou

Memorial University of Newfoundland, Canada

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(with Tsiu-Kwen Lee)

Clean rings

R: associative ring with 1
$U(R)$: group of units of R
$J(R)$: Jacobson radical of R
$E(M)$: injective hull of a module M

- An element $a \in R$ is **clean** if $\exists e^2 = e$ and $u \in U(R)$ such that $a = e + u$, and R is **clean** if every $a \in R$ is clean. [Nicholson, 77]

- Clean rings form a proper subclass of exchange rings. [Nicholson, 77], [Camillo-Yu, 94]

  (R is an **exchange ring** $\Leftrightarrow \forall a \in R, \exists e^2 = e \in aR$ with $1 - e \in (1-a)R$.)

  Bergman's example:

  Let K be a field , $A = K[[x]]$, Q the field of fractions of A. Define

$$R = \{r \in \mathrm{End}(A_K) : \exists q \in Q \text{ and } \exists n > 0 \text{ with } r(a) = qa \text{ for all } a \in x^n A\}.$$

  Then R is a regular (so exchange) ring. But if $\mathrm{char}(K) \neq 2$, then R is not clean.

- **Examples** of clean rings:
  1. Semiperfect rings. [Camillo-Yu, 94]
  2. Unit-regular rings. [Camillo-Khurana, 01]
  3. Exchange rings whose idempotents are central. [Nicholson, 77]
  4. Strongly π-regular rings (i.e. for $\forall a \in R \ \exists n \geq 1$ such that $a^n R = a^{n+1} R$). [Burgess-Menal, 88]
  5. End(M_R) where M_R is either continuous or discrete or flat cotorsion. [Camillo-Khurana-Lam-Nicholson-Z, 06]
  6. If R is clean then $\mathbb{M}_n(R)$, $\mathbb{T}_n(R)$ and $R[[t]]$ are all clean. But $R[t]$ is not clean for any ring R (because t is not the sum of a unit and an idempotent).
  7. ......

Uniquely clean rings

- An element $a \in R$ is uniquely clean if $\exists | e^2 = e$ and $\exists | u \in U(R)$ such that $a = e + u$, and R is called **uniquely clean** if every $a \in R$ is uniquely clean.

- **Structure Theorem** [Nicholson-Z, 04]

  1. R is local, uniquely clean iff $R/J(R) \cong \mathbb{Z}_2$.
  2. R is semiprimitive, uniquely clean iff R is Boolean.
  3. R is uniquely clean iff $R/J(R)$ is Boolean, idempotents of R are central, and idempotents lift modulo $J(R)$.

- **Motivated question**:

  Establish the structure for a larger class of clean rings including uniquely clean rings.

Generalizations of Boolean rings

- For a prime p, a p-**ring** is a ring R in which $a^p = a$ and $pa = 0$ for all $a \in R$.

  R is a p-ring iff it is a subdirect product of fields of order p. [N.H.McCoy - D.Montgomery, 37]

- For a prime p and a positive integer k, a p^k-**ring** is a ring R in which $a^{p^k} = a$ and $pa = 0$ for all $a \in R$.

  A p^k-ring is isomorphic to the ring of continuous functions (with an extra condition) from a locally compact zero-dimensional space to the Galois field of p^k elements. [R.Arens - I.Kaplansky, 48]

- For a positive integer n, a $J(n)$-**ring** is a ring R such that $a^{n+1} = a$ for all $a \in R$. A ring R is called a J-**ring** if R is a $J(n)$-ring for some $n \geq 1$.

  R is a J-ring iff it is the direct sum of finitely many p^k-rings. [J.Luh, 67]

- A ring R is **periodic** if for each $a \in R$ there is a positive integer $n(a)$ such that $a^{n(a)+1} = a$.

  R is periodic iff it is the union of a countable ascending chain $\{R_i\}$ of J-rings such that every J-ring contained in R is contained in some R_i.
  [T.Chinburg - M.Henriksen, 76]

Structure Theorems

- Let $n > 0$. A ring R is called **uniquely n-clean** if a^n is uniquely clean for every $a \in R$. The ring R is called **uniquely π-clean** if for each $a \in R$ there exists a positive integer $n(a)$ such that $a^{n(a)}$ is uniquely clean.

- R is uniquely π-clean iff $R/J(R)$ is a periodic ring, idempotents of R are central and idempotents lift modulo $J(R)$.

- R is uniquely n-clean iff $R/J(R)$ is a $J(n)$-ring, idempotents of R are central and idempotents lift modulo $J(R)$.

  Letting $n = 1$ yields

  Coro. [Nicholson-Z, 04] R is uniquely clean iff $R/J(R)$ is Boolean, idempotents of R are central, and idempotents lift modulo $J(R)$.

- Some arguments of the Proof:

  1. If a^n is strongly clean for some $n \geq 1$, then a is strongly clean.
     (a^n clean $\stackrel{?}{\Rightarrow}$ a clean)
  2. For $e^2 = e \in R$, e is uniquely clean iff e is central.
  3. Let R be a uniquely π-clean ring. Then $u \in U(R)$ is uniquely clean iff $1 - u \in J(R)$.

Structure Theorems (continued)

- A ring R is a uniquely π-clean iff R is the union of a countable ascending chain $\{R_i\}$ of subrings where, for each i, $R_i \supseteq J(R)$ is a uniquely n_i-clean ring for some $n_i \geq 1$ such that any uniquely n-clean ring contained in R is contained in some R_i.

- TFAE for a ring R:

  1. R is a uniquely n-clean ring for some $n \geq 1$.

  2. $R = R_1 \oplus \cdots \oplus R_s$, where R_i is a uniquely $(p_i^{k_i} - 1)$-clean ring and $p_i R_i \subseteq J(R_i)$ with p_i a prime and $k_i \geq 1$ for $i = 1, \ldots, s$.

Examples

- Let R be a local ring and $n \geq 1$. Then R is uniquely n-clean iff $R/J(R) \cong GF(p^k)$ where p is a prime and $k \geq 1$ such that $(p^k - 1) \mid n$ and where $GF(p^k)$ denotes the Galois field of p^k elements.

- Let σ be an endomorphism of R and $n \geq 1$. Then $R[[x;\sigma]]$ is uniquely n-clean (resp., uniquely π-clean) iff R is uniquely n-clean (resp., uniquely π-clean) and $\sigma(e) = e$ for all $e^2 = e \in R$.

- Let n, m be positive integers. Then R is uniquely n-clean (resp., uniquely π-clean) iff $R[x]/(x^m)$ is uniquely n-clean (resp., uniquely π-clean).

- Every factor ring of a uniquely n-clean (resp. uniquely π-clean) ring is uniquely n-clean (resp. uniquely π-clean).

Examples as group rings

- Let R be a ring, G a group, p a prime and $k \geq 1$. Then the following hold:

  1. If RG is uniquely $(p^k - 1)$-clean, then R is uniquely $(p^k - 1)$-clean and, for any $g \in G$, $o(g) = p^s q$ where $s \geq 0$ and $q \mid (p^k - 1)$.
  2. If R is uniquely $(p^k - 1)$-clean and G is a locally finite p-group, then RG is uniquely $(p^k - 1)$-clean.

  Letting $p = 2$ and $k = 1$ yields

  Coro. [Chen-Nicholson-Z, 06] If the group ring RG is uniquely clean, then R is a uniquely clean ring and G is a 2-group. The converse holds if G is locally finite.

  Coro. Let R be a ring and let G be an abelian group. Then RG is uniquely 2-clean iff R is uniquely 2-clean and G is the direct product of a 3-group and an elementary 2-group.

Conditions on a module M

- M is **CS** if it satisfies

  (C_1) Every submodule of M is essential in a summand of M.

- M is **continuous** if it satisfies (C_1) and

  (C_2) Every submodule of M that is isomorphic to a summand of M is itself a summand of M.

- M is **quasi-continuous** if it satisfies (C_1) and

  (C_3) If A, B are summands of M with $A \cap B = 0$, then $A \oplus B$ is also a summand of M.

- M is **quasi-injective** if every homomorphism from any submodule of M to M extends to an endomorphism of M.

- Quasi-injective $\Rightarrow$ continuous $\Rightarrow$ quasi-continuous $\Rightarrow$ CS; none of the arrows is reversible.

A consequence of

Theorem. If M_R is a continuous module, then End(M_R) is a clean ring. [Camillo-Khurana-Lam-Nicholson-Z, 06]

- M is quasi-injective $\Leftrightarrow$ $\sigma M \subseteq M$, $\forall \sigma \in$ End$(E(M)_R)$. [R.E. Johnson - E.T. Wong, 61]

- M is quasi-injective $\Leftrightarrow$ $\sigma M \subseteq M$, $\forall \sigma^2 = \sigma \in$ End$(E(M)_R)$ and $\forall \sigma \in$ Aut$(E(M)_R)$. (by the theorem)

- M is quasi-continuous $\Leftrightarrow$ $\sigma M \subseteq M$, $\forall \sigma^2 = \sigma \in$ End$(E(M)_R)$. [L. Jeremy, 74]

- ??? $\Leftrightarrow$ $\sigma M \subseteq M$, $\forall \sigma \in$ Aut$(E(M)_R)$.

Automorphism-invariant modules

- A module M is called an **automorphism-invariant module** (or auto-invariant module) if $\sigma M \subseteq M$ for every automorphism σ of $E(M)$.

- quasi-injective = auto-invariant + quasi-continuous

- Examples of auto-invariant modules:

  quasi-injective modules and, more generally, pseudo-injective modules.

  M is **pseudo-injective** if every monomorphism from a submodule of M to M extends to an endomorphism of M.
  [S. Singh - S.K. Jain, 67]

A characterization

- TFAE for a module M:
  1. M is an auto-invariant module.
  2. Every isomorphism between two essential submodules of M extends to an endomorphism of M.
  3. Every isomorphism between two essential submodules of M extends to an automorphism of M.

Direct sums

- $M_1 \oplus M_2$ is quasi-continuous iff each summand is quasi-continuous and M_1, M_2 are relatively injective. [Müller-Rizvi, 83]

- If $M_1 \oplus M_2$ is auto-invariant, then each summand is auto-invariant and M_1, M_2 are relatively injective.

  Coro. M is quasi-injective iff $M \oplus M$ is auto-invariant.

  Coro. R is semisimple Artinian iff every 2-generated R-module is auto-invariant.

  Coro. [Dinh, 05] If $M_1 \oplus M_2$ is pseudo-injective, then M_1, M_2 are relatively injective.

Dinh's question

- Every pseudo-injective module satisfies (C_2), so every pseudo-injective CS module is continuous. [Dinh, 05]

  Dinh's question: Is a pseudo-injective CS module quasi-injective?

- M is quasi-injective iff M is pseudo-injective and $M \oplus M$ is CS. [Alahmadi, Er and Jain, 05]

- M is quasi-injective iff M is pseudo-injective and M is CS. [Ganesan-Vanaja,07]

  Proof.

  Quasi-injective $\Rightarrow$pseudo-injective + CS

  $\Rightarrow$pseudo-injective + quasi-continuous (by Dinh's theorem)

  $\Rightarrow$auto-invariant+quasi-continuous

  $=$quasi-injective (by our observation)

Auto-invariant + CS = quasi-injective

- Every auto-invariant module satisfies (C_3).

  Proof. Let M be an auto-invariant module. Assume that A, B are two summands of M such that $A \cap B = 0$. We need to show that $A \oplus B$ is a summand of M. Write $M = A \oplus A'$, and let $\pi : M \to A'$ be the canonical projection. Let C be a submodule of M such that $(A+B) \cap C = 0$ and $A \oplus B \oplus C \leq_e M$. Write $D := B \oplus C$. Then $A \oplus D = A \oplus \pi D$, and $\pi|_D : D \to \pi D$ is an isomorphism. Thus $1_A \oplus \pi|_D : A \oplus D \to A \oplus \pi D$ is an isomorphism. Since M is auto-invariant and $A \oplus D$ is essential in M, $1_A \oplus \pi|_D$ extends to an automorphism σ of M. Since B is a summand of M, $\pi B = \sigma B$ is a summand of M and so πB is a summand of A'. Hence $A \oplus B = A \oplus \pi B$ is a summand of M.

- M is quasi-injective iff it is auto-invariant CS.

  Proof.

  Quasi-injective =auto-invariant+quasi-continuous
  (by our observation)
  =auto-invariant + CS
  (by the result above)

Over a semiprime right Goldie ring

- Over a semiprime right Goldie ring, every nonsingular quasi-injective module is injective. [Boyle-Goodearl, 75]

- Over a prime right Goldie ring, every nonsingular pseudo-injective module is injective. [Jain-Singh, 75]

- Over a semiprime right Goldie ring, every nonsingular auto-invariant module is injective.

Questions and remarks

Any decomposition of an auto-invariant module? When is a direct sum of modules auto-invariant? The endomorphism ring of an auto-invariant module? A ring R is a right QI-ring if every quasi-injective right R-module is injective. Which rings R have the property that every auto-invariant right R-module is injective? ....

THANK YOU