On Frobenius algebras and Frobenius monads

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Overview

- Frobenius algebras
- Monads and comonads
- Frobenius monads
- Frobenius and quasi-Frobenius functors

Frobenius algebras

Matrix rings over field K

$A = \mathrm{Mat}_n(K)$ matrix ring, trace $\mathrm{Tr} : A \to K$

$$\text{bilinear form} \quad \beta : A \otimes_K A \xrightarrow{\mu} A \xrightarrow{\mathrm{Tr}} K,$$

$$\beta(a, b) = \mathrm{Tr}(ab),$$

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bilinear form	$\beta : A \otimes_K A \xrightarrow{\mu} A \xrightarrow{\mathrm{Tr}} K,$
	$\beta(a, b) = \mathrm{Tr}(ab),$
associative	$\beta(ab, c) = \beta(a, bc),$
symmetric	$\beta(a, b) = \beta(b, a),$
non-degenerate	$A \to A^*, \; a \mapsto \beta(a, -)$, is A-isomorphism.

Frobenius algebras

Frobenius algebras over K

A K-algebra, A_K finite dimensional; equivalent

(a) linear form $t : A \to K$,
$\mathrm{Ke}\, t$ contains no non-zero (left) ideal of A;

(b) bilinear form $\beta : A \otimes_K A \longrightarrow K$,
associative and nondegenerate;

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(b) bilinear form $\beta : A \otimes_K A \longrightarrow K$,
associative and nondegenerate;

(c) A-isomorphism $\lambda : A \to A^*$ $(a \mapsto \beta(a, -))$.

Frobenius algebras

Coalgebra structure of A^*

Multiplication and unit on A:

$$\mu : A \otimes_K A \longrightarrow A, \quad \eta : K \longrightarrow A,\ k \mapsto k1_A.$$

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Apply $(\ )^* = \mathrm{Hom}_K(-, K)$, comultiplication and counit on A^*:

$$A^* \xrightarrow{\mu^*} (A \otimes_K A)^* \simeq A^* \otimes_K A^*, \quad A^* \xrightarrow{\eta^*} K.$$

Frobenius algebras

Coalgebra structure on A (Lowell Abrams, 1999), $\lambda : A \to A^*$

$$\begin{array}{ccc} A & \xrightarrow{\delta} & A \otimes_K A \\ {\scriptstyle\lambda}\downarrow & & \uparrow{\scriptstyle \lambda^{-1}\otimes\lambda^{-1}} \\ A^* & \xrightarrow{\mu^*} & A^* \otimes_K A^* \end{array} \quad , \quad \varepsilon := \lambda(1_A) : A \to K.$$

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satisfies Frobenius conditions

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\mu} & A \\ \big\downarrow{\scriptstyle I\otimes\delta} & & \big\downarrow{\scriptstyle \delta} \\ A \otimes A \otimes A & \xrightarrow{\mu\otimes I} & A \otimes A \end{array} \quad , \quad \begin{array}{ccc} A \otimes A & \xrightarrow{\mu} & A \\ \big\downarrow{\scriptstyle \delta\otimes I} & & \big\downarrow{\scriptstyle \delta} \\ A \otimes A \otimes A & \xrightarrow{I\otimes\mu} & A \otimes A \end{array}$$

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Theorem: A-modules are A-comodules: $\mathbb{M}_A \simeq \mathbb{M}^A$

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(2) $\mu : A \otimes_K A \to A, \quad \eta : K \to A$ induce natural transformation

$$A \otimes_K A \otimes_K - \to A \otimes_K -, \quad K \otimes_K - \to A \otimes_K -;$$

(3) $A \otimes_K -$ has a right adjoint $\mathrm{Hom}_K(A, -)$,

$$\mathrm{Hom}_K(A \otimes_K M, N) \simeq \mathrm{Hom}_K(M, \mathrm{Hom}_K(A, N));$$

(4) dim $A_K < \infty$: $\mathrm{Hom}_K(A, -)$ and $A^* \otimes_K -$ isomorphic by

$$A^* \otimes_K M \mapsto \mathrm{Hom}_K(A, M), \; f \otimes m \mapsto [a \mapsto f(a)m];$$

(5) Frobenius condition: $A \otimes_K - \simeq \mathrm{Hom}_K(A, -)$.

General categories

Monads on $\mathbb{A}$

$\mathbf{F} = (F, \mu, \eta)$, where $F : \mathbb{A} \to \mathbb{A}$ is a functor with natural transformations

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satisfying certain commutative diagrams (as for algebras).

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objects $A \in \mathrm{Obj}(\mathbb{A})$ with morphisms $\quad \varrho : F(A) \to A$
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Free functor $\phi_F : \mathbb{A} \to \mathbb{A}_F,\ A \mapsto [F(A), FF(A) \xrightarrow{\mu_A} F(A)],$

with right adjoint forgetful functor $U_F : \mathbb{A}_F \to \mathbb{A}$.

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Comonad on $\mathbb{A}$

$\mathbf{G} = (G, \delta, \varepsilon)$, where $G : \mathbb{A} \to \mathbb{A}$ is a functor with natural transformations

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Adjoint endofunctors $F : \mathbb{A} \to \mathbb{A}$, $G : \mathbb{A} \to \mathbb{A}$

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\mathrm{Mor}_{\mathbb{A}}(FF(X), Y) & & \vdots\; ? \\
\downarrow{\scriptstyle \varphi_{F(X),Y}} & & \downarrow \\
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implies G comonad

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(F, G) adjoint with unit $u : I \to GF$ and counit $e : FG \to I$. Then

$$\text{every } F\text{-module} \quad \rho : F(A) \to A$$

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Theorem (Eilenberg-Moore 1965)

The functor

$$\mathbb{A}_F \to \mathbb{A}^G, \quad F(A) \xrightarrow{\rho} A \longmapsto A \xrightarrow{u_A} GF(A) \xrightarrow{G\rho} G(A)$$

induces an isomorphism of categories.

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(a) F is a Frobenius monad;

(b) F has comonad structure $\overline{F} = (F, \delta, \varepsilon)$ with isomorphism

$K : \mathbb{A}_F \to \mathbb{A}^F$ such that

$$\begin{array}{ccc} \mathbb{A}_F & \xrightarrow{\quad K \quad} & \mathbb{A}^F \\ & {\scriptstyle U_F} \searrow \quad \swarrow {\scriptstyle U^F} & \\ & \mathbb{A} & \end{array}$$

is commutative.

Adjoint endofunctors of ${}_R\mathbb{M}$

$A \otimes_R -, \quad \mathrm{Hom}_R(A,-) \;:\; {}_R\mathbb{M} \to {}_R\mathbb{M}$

$$\mathrm{Hom}_R(A \otimes_R X, Y) \xrightarrow{\simeq} \mathrm{Hom}_R(X, \mathrm{Hom}_R(A, Y)).$$

Equivalent for $A \in {}_R\mathbb{M}$:

(a) $A \otimes_R - : {}_R\mathbb{M} \to {}_R\mathbb{M}$ is a monad $\quad$ (A is an R-algebra);

(b) $\mathrm{Hom}_R(A,-) : {}_R\mathbb{M} \to {}_R\mathbb{M}$ is a comonad.

Correspondence

$$\mu : A \otimes_R A \to A, \qquad u : R \to A,$$

correspond to

$$\mathrm{Hom}_R(A,-) \xrightarrow{\mu^*} \mathrm{Hom}_R(A \otimes_R A, -) \xrightarrow{\simeq} \mathrm{Hom}_R(A, \mathrm{Hom}_R(A,-)),$$
$$\mathrm{Hom}_R(A,-) \xrightarrow{u^*} \mathrm{Hom}_R(R,-).$$

Modules and comodules in ${}_R\mathbb{M}$

A-modules are $\mathrm{Hom}_R(A,-)$-comodules

$\rho_N : A \otimes_R N \to N$ induces

$$\widehat{\rho_N} : N \xrightarrow{\nu_N} \mathrm{Hom}_R(A, A \otimes_R N) \xrightarrow{\mathrm{Hom}(A,\rho_N)} \mathrm{Hom}_R(A, N).$$

$\rho^N : N \to \mathrm{Hom}_R(A, N)$ induces

$$\widetilde{\rho^N} : A \otimes_R N \xrightarrow{I \otimes \rho^N} A \otimes_R \mathrm{Hom}_R(A, N) \xrightarrow{\varepsilon_N} N.$$

Equivalence

$$\mathbb{M}_A \to \mathbb{M}^{[A,-]}, \qquad \begin{array}{rl} (M, \rho_N) & \mapsto (M, \widehat{\rho_N}) \\ f : M \to N & \mapsto f : M \to N. \end{array}$$

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Any Frobenius functor $F : \mathbb{A} \to \mathbb{B}$ preserves all limits and colimits which exist in $\mathbb{A}$.

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Proposition

A monad F on $\mathbb{A}$ is a *Frobenius monad* if and only if the forgetful functor $U_F : \mathbb{A}_F \longrightarrow \mathbb{A}$ is Frobenius.

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An R-algebra A is a Frobenius extension if and only if the forgetful functor $\mathbb{M}_A \to \mathbb{M}_R$ is Frobenius.

General categories

Adjunction context

$L : \mathbb{A} \to \mathbb{B}$, $R : \mathbb{B} \to \mathbb{A}$ functors, morphisms, natural in A and B,

$$\alpha_{A,B} : \mathrm{Mor}_{\mathbb{B}}(L(A), B) \to \mathrm{Mor}_{\mathbb{A}}(A, R(B)),$$
$$\beta_{A,B} : \mathrm{Mor}_{\mathbb{A}}(A, R(B)) \to \mathrm{Mor}_{\mathbb{B}}(L(A), B).$$

$$\eta_A := \alpha_{A,L(A)}(I) : A \to RL(A)$$
$$\varepsilon_B := \beta_{R(B),B}(I) : LR(B) \to B$$

natural transformations $\quad \eta : I_{\mathbb{A}} \to RL, \quad \varepsilon : LR \to I_{\mathbb{B}}.$

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The context is called

left semi-adjoint	if $\beta \circ \alpha = I$,
right semi-adjoint	if $\alpha \circ \beta = I$,
an adjunction	left and right semi-adjoint.

General categories

Quasi Frobenius: $F : \mathbb{A} \longrightarrow \mathbb{B}$ is called

right quasi-Frobenius	has right adjoint functor $R : \mathbb{B} \longrightarrow \mathbb{A}$ and (R^{Λ}, F) is right semi-adjoint, some Λ;
left quasi-Frobenius	has left adjoint functor $L : \mathbb{B} \longrightarrow \mathbb{A}$ and $(F, L^{(\Lambda')})$ is left semi-adjoint, some Λ';
quasi-Frobenius	left and right quasi-Frobenius.

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$F : \mathbb{A} \longrightarrow \mathbb{B}$ functor, left adjoint L, right adjoint R

Equivalent: (a) (L, F) is left quasi-Frobenius;
(b) R is a retract of $L^{(\Lambda)}$.

Then R preserves all colimits which exist in $\mathbb{B}$.

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Then	L preserves all limits which exist in $\mathbb{B}$.

General categories

Definition

A monad (F, μ, η) on $\mathbb{A}$ is a *quasi-Frobenius monad* if the forgetful functor $U_F : \mathbb{A}_F \longrightarrow \mathbb{A}$ is a quasi-Frobenius functor.

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Properties of quasi-Frobenius monads

(i) ϕ_F and ϕ^G preserve all limits and colimits;

(ii) ϕ_F preserves small objects;

(iii) every object in $\mathbb{A}_F$ is embedded in an U_F-injective and factor of a U_F-projective object;

(iv) the U_F-injective objects in $\mathbb{A}_F$ are the same as the U_F-projectives.

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QF-rings

A ring R is a QF-ring: every injective R-module is projective.

References

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