A Matrix Representation of an Azumaya Group Ring

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Abstract

Let R be an indecomposable ring with 1 of characteristic p^k for some prime integer p and integer k, G a group, and RG a group ring of G over R. It is shown that if RG is an Azumaya algebra, then RG contains a direct sum of matrix rings over Azumaya algebras, and RG is a direct sum of matrix rings over Azumaya algebras if and only if the center of RCG' is C where C is the center of RG and G' is the commutator subgroup of G.

(I) Definitions and Notations

(1) Let B be a ring with 1 and A a subring of B with the same identity 1. Then B is called a separable extension of A if there exist $\{a_i, b_i$ in B, $i = 1, 2, ..., m$ for some integer $m\}$ such that $\sum a_i b_i = 1$, and $\sum x a_i \otimes b_i = \sum a_i \otimes b_i x$ for all x in B where $\otimes$ is over A.

(2) Let C be the center of B. If B is a separable extension over C, then B is called an Azumaya C-algebra.

(3) A split Azumaya C-algebra B is an Azumaya C-algebra such that $B \cong Hom_C(P, P)$ where P is a progenerator of C.

For more about separable algebras and Azumaya algebras, see [1].

(II) Introduction

(1) **Theorem 1** ([7])

If R is a ring of p-adic integers and G a semidirect product of a finite abelian p-group and a finite cyclic group of order not divisible by p, then RG is a direct sum of matrix rings over commutative rings.

(2) **Theorem 2** ([2]) (A generalization of Theorem 1)

Let RG be an Azumaya group algebra, R a complete local ring with residue field of characteristic p. Then RG is a direct sum of matrix rings over commutative rings if and only if G is a semidirect product of an abelian p-Sylow subgroup with a normal p-complement.

(3) **Theorem 3** ([5]) (The general Brauer splitting theorem)

Let R be an indecomposable commutative ring with 1, G a group of order n invertible in R for some integer n, and θ a primitive $n^{\mathbf{th}}$ root of 1. Then $(R[\theta])G$ is a splitting Azumaya group algebra.

(4) **Theorem 4** ([2]) (A characterization of an Azumay group ring)

Let R be a ring with 1, G a group, and RG a group ring of G over R. Then RG is an Azumaya algebra if and only if (1) R is an Azumaya algebra, (2) the center of G has a finite index, and (3) the order of the commutator subgroup G' of G is a finite integer and invertible in R.

(III) Main Results

We shall give an equivalent condition under which an Azumaya group ring RG is a direct sum of matrix rings over Azumaya algebras, where R is an indecomposable ring of characteristic p^k for some prime integer p and an integer k.

We shall employ the general Wedderburn theorem ([3], Corollary 1).

Proposition 3.1.

Let A be an Azumaya algebra over an indecomposable semi-local ring C. Then $A \cong M_n(D)$, a matrix ring of order n for some integer n over an indecomposable Azumaya algebra D unique up to isomorphism.

Lemma 3.2.

Let R be an indecomposable ring of characteristic p^k, G a group, and G' the commutator subgroup of G. If RG is an Azumaya algebra over its center C, then RG' is a finite direct sum of matrix rings $M_{n_i}(E_i)$ of order n_i for some integer n_i over an Azumaya algebra E_i.

Proof. (Outline)
$\mathrm{Char}(R) = p^k \Longrightarrow$, the prime ring of R_0 of $R \cong Z/Zp^k$.
RG is an Azumaya C-algebra $\Longrightarrow |G'|$ is finite and invertible in R
$\Longrightarrow R_0G'$ is separable group algebra. Let C_0 be the center of R_0G'.
Then C_0G is an Azumaya C_0-algebra and C_0 is a finite direct sum of indecomposable semi-local rings, i.e., $C_0 \cong \oplus \sum_{i=1}^{l} C_0e_i$
where $\{e_i\}$ are minimal idempotents of $C_0 \Longrightarrow$
$RG' \cong R \otimes_{R_0} R_0G' \cong R \otimes_{R_0} (\oplus \sum_{i=1}^{l} R_0e_iG')$
$\cong \oplus \sum_{i=1}^{l}(R \otimes_{R_0} M_{n_i}(D_i)) \cong \oplus \sum_{i=1}^{l} M_{n_i}(R \otimes_{R_0} D_i)$
by Proposition 3.1. RG is Azumaya $\Longrightarrow$ so is $R \Longrightarrow$ so is $R \otimes_{R_0} D_i$.
Let $E_i = R \otimes_{R_0} D_i$ by E_i. Then $RG' \cong \oplus \sum_{i=1}^{l} M_{n_i}(E_i)$.

Proposition 3.3. (The commutator theorem for Azumaya algebras)

Let A be an Azumaya algebra and B a separable subalgebra of A. Then the commutator subalgebra A^B of B in A is also a separable subalgebra of A such that $B \otimes_{C'} A^B \cong B \cdot A^B$ where C' is the center B. In particular, if $C = C'$, then $B \otimes_C A^B \cong B \cdot A^B = A$

Theorem 3.4.

Let RG be given in Lemma 3.2. Then RG contains a separable subalgebra $(RG')(R_0G)^{G'} \cong \oplus \sum M_{n_i}(E_i \otimes_{C'} (R_0G)^{G'})$ where C' is the center of E_i.

Proof. This a a consequence of Lemma 3.2 and Proposition 3.3.

Theorem 3.5.

Let RG be given in Theorem 3.4. Then the following are equivalent: (1) $RG \cong \oplus \sum_i M_{n_i}(E_i \otimes_{C'} (R_0G)^{G'})$; (2) $C =$ the center of RCG'; (3) $C' \subset C$ where C' is the center of E_i (= the center of RG').

Proof. (1)$\Longleftrightarrow$(2) By Theorem 3.4, RG' is an Azumaya algebra over C' and RCG' is a separable subalgebra of RG over C, so the center C'' of RCG' is $C'C$; that is, $C'' = C'C$. Thus, by the commutator theorem for Azumaya algebras, $RG \cong \oplus \sum_i M_{n_i}(E_i \otimes_{C'} (R_0G)^{G'})$ if and only if $C' \subset C$.

(2)$\Longleftrightarrow$(3) is clear.

Let RG be an Azumaya group algebra over an indecomposable commutative ring R of characteristic p^k. We shall show (1) $(R[\theta])G'$ is a direct sum of matrix rings over commutative ring where θ is a primitive $m^{\mathbf{th}}$ root of 1 and m is the order of G', and (2) If $(RG)^{G'}$ is a direct sum of matrix rings over commutative rings, then so is $(R[\theta])G$.

Lemma 4.1.

Let R be a commutative indecomposable ring, G a finite group of order n invertible in R. Then $(R[\theta])G \cong \oplus \sum_{i=1}^{l} M_{n_i}(R[\theta])$ for some integer l where θ is a primitive $n^{\mathbf{th}}$ root of 1.

Theorem 4.2.

Let R be a commutative indecomposable ring of characteristic p^k. Assume RG is an Azumaya algebra. If $((R[\theta])G)^{G'}$ is a direct sum of matrix rings over commutative rings and $C =$ the center of $(R[\theta]C)G'$, then $(R[\theta])G$ is a direct sum of matrix rings over commutative rings.

Theorem 4.3.

Let R be a commutative indecomposable ring of characteristic p^k. Assume RG is an Azumaya algebra. Then $(R[\theta])G$ is a splitting Azumaya algebra if and only if so is $(R[\theta]G)^{G'}$.

References

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