SYMMETRIES ON BDD OBSERVABLES

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H Hilbert space

$\mathcal{S}(H)$, $\mathcal{E}(H) = \{A \in \mathcal{S}(H) : 0 \le A \le I\}$

Symmetries:

$$\phi : \mathcal{S}(H) \to \mathcal{S}(H)$$

$$\mathcal{E}(H) \to \mathcal{E}(H) \quad \text{bijective}$$

$$A \leftrightarrow B \iff \phi(A) \leftrightarrow \phi(B)$$

$$A \le B \iff \phi(A) \le \phi(B)$$

$$\phi(ABA) = \phi(A)\phi(B)\phi(A)$$

$\vdots$

$$\phi : \mathcal{S}(H) \to \mathcal{S}(H) \quad \text{bij.}$$

$$AB = BA \iff \phi(A)\phi(B) = \phi(B)\phi(A)$$

$$A \leftrightarrow B \implies A^2 \leftrightarrow e^B$$

$$\phi(A) = U f_A(A) U^*$$

$$\phi : \mathcal{S}(H) \to \mathcal{S}(H) \quad \text{bij}$$

$$A \leq B \iff \phi(A) \leq \phi(B)$$

$$\Downarrow$$

$$\phi(A) = TAT^* + S$$

PR. $\phi : \mathcal{E}(H) \to \mathcal{E}(H)$ bijective

$$A \leq B \iff \phi(A) \leq \phi(B)$$

LUDWIG : $A^{\perp} = I - A$

$$\phi(A^{\perp}) = \phi(A)^{\perp}$$

$$\Downarrow$$

$$\phi(A) = UAU^*$$

MOLNÁR : 1) coexistency $\Rightarrow \phi(A) = UAU^*$

2) $AB = 0 \iff \phi(A)\phi(B) = 0$

$$\Downarrow$$

$$\phi(A) = U f_p(A) U^*$$

$$f_p(x) = \frac{x}{px + (1-p)}$$

$$f_p : [0,1] \to [0,1] \quad \text{bij.}$$

$$A, B \in \mathcal{E}(H) \ \& \ A \leq B \Rightarrow f_p(A) \leq f_p(B)$$

3) $\exists \lambda, \mu \in (0,1) : \phi(\lambda I) = \mu I$

$$\Downarrow$$

$$\phi(A) = U f_p(A) U^*$$

$$A \mapsto \left(\frac{T^2}{2I - T^2}\right)^{-1/2} \left(\left(I - T^2 + T(I+A)^{-1}T\right)^{-1} - I \right) \left(\frac{T^2}{2I - T^2}\right)^{-1/2}$$

$\vdots$

$A, B \in \mathcal{S}(H)$

A adj. to $B \iff \operatorname{rank}(B-A)=1$

A adj. to $B \Rightarrow$

$B = A + tP \Rightarrow$

$A \le B$ or $B \le A$

$\Downarrow$

$[A,B] = \{ A + DP : 0 \le D \le t \}$

$\Downarrow$

$C, D \in [A,B] \Rightarrow C \le D$ or $D \le C$

ϕ: bijective & preserves order (comp.)

$\Downarrow$

ϕ preserves adjacency

A, B adj. $\Leftrightarrow$ $\phi(A), \phi(B)$ adj.

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line $\{A + tP : t \in \mathbb{R}\}$

$\uparrow$

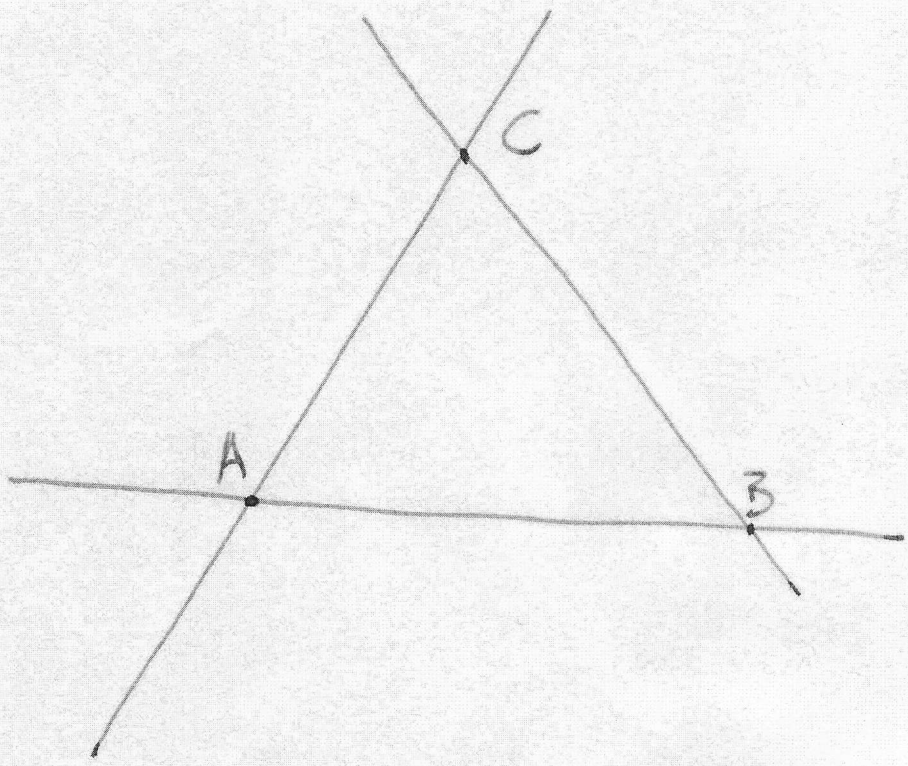
maximal adjacent set

lines $\xrightarrow{\phi}$ lines

~~FTAF~~

geometry : lines, points

no triangles



WLOG: $A = 0$

$B = \lambda P$

$C = \mu Q$

$$M_4 = \{(x,y,z,t) : x,y,z,t \in \mathbb{R}\}$$

(x_1, y_1, z_1, t_1) coherent (x_2, y_2, z_2, t_2)

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = c^2(t_2 - t_1)^2$$

$$A_1 = \begin{bmatrix} ct_1 + z_1 & x_1 + iy_1 \\ x_1 - iy_1 & ct_1 - z_1 \end{bmatrix} \qquad A_2$$

$$\det(A_2 - A_1) = c^2(t_2 - t_1)^2 - (z_2 - z_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2$$

$A < B$

$[A,B] = \{C \in \mathcal{S}(H) : A \leq C \leq B\}$

$\phi : [A,B] \to [C,D]$ bijective

ϕ order automf:

$$A \leq B \iff \phi(A) \leq \phi(B)$$

ϕ order anti-automf:

$$A \leq B \iff \phi(B) \leq \phi(A)$$

Examples:

$$[A,B] \longrightarrow [A+C, B+C]$$

$$T \mapsto T+C$$

$$[A,B] \longrightarrow [CAC^*, CBC^*]$$

$$T \mapsto CTC^*$$

$$0 < A < B$$

$$[A,B] \longrightarrow [B^{-1}, A^{-1}]$$

$$T \mapsto T^{-1}$$

$$0 < E \leq F \Rightarrow F^{-1/2} E F^{-1/2} \leq I \Rightarrow I \leq F^{1/2} E^{-1} F^{1/2}$$

$$\Rightarrow F^{-1} \leq E^{-1}$$

$$A \mapsto \left(\frac{T^2}{2I-T^2}\right)^{-1/2}\left(\left(I-T^2+T(I+A)^{-1}T\right)^{-1}-I\right)\left(\frac{T^2}{2I-T^2}\right)^{-1/2}$$

$$A \mapsto I+A \mapsto (I+A)^{-1} \mapsto T(I+A)^{-1}T$$

$$\mapsto I-T^2+T(I+A)^{-1}T \mapsto \left(I-T^2+T(I+A)^{-1}T\right)^{-1}$$

$$\mapsto \left(I-T^2+T(I+A)^{-1}T\right)^{-1}-I$$

$$0 \mapsto 0$$

$$I \mapsto T^2(2I-T^2)^{-1}$$

$r > 0$

$$A \mapsto r(r+1)\left[(rI+A)^{-1} - \frac{1}{r+1} I\right] = \delta(A)$$

order-antiautom. of $E(H)$

$$\delta(A) = g_r(A)$$

$$g_r(x) = r\,\frac{1-x}{r+x}$$

$$g_s(g_r(x)) = \frac{x}{px+(1-p)}, \quad x \in [0,1]$$

$$p = \frac{s-r}{s(1+r)}$$

TH : $\phi : \mathcal{E}(H) \to \mathcal{E}(H)$ bij.

$$A \leq B \iff \phi(A) \leq \phi(B)$$

$$\Downarrow$$

$\exists\, p, q \in (-\infty, 1)$

$T : H \to H$ bij., bounded, $1/2$ - lin.

$\|T\| \leq 1$

$$\phi(A) = f_q\left(\left(f_p(TT^*)\right)^{-1/2} f_p(TAT^*) \left(f_p(TT^*)\right)^{-1/2} \right)$$

$$\phi : \mathcal{E}(H) \to \mathcal{E}(H) \quad \text{bij.}$$

$$A \text{ comp. } B \iff \phi(A) \text{ comp } \phi(B)$$

More examples: $A \mapsto A^{\perp} = I - A$

$$A \mapsto \begin{cases} A & : A \neq 0, I \\ I & : A = 0 \\ 0 & : A = I \end{cases}$$

BACK TO LAJOS:

$$\phi : \mathcal{E}(H) \to \mathcal{E}(H) \quad \text{bij}$$

$$A \leq B \iff \phi(A) \leq \phi(B)$$

$$\exists \lambda, \mu \in (0,1) : \phi(\lambda I) = \mu I$$

$$\Downarrow$$

$$\phi(A) = U f_p(A) U^*$$

Step 1 $\quad \phi : (0, I) \to (0, I)$

$$\Downarrow$$

$$\phi(\tfrac{1}{2} I) = R \in (0, I)$$

$$\xi(R) = \tfrac{1}{2} I$$

$$(\xi \circ \phi)(\tfrac{1}{2} I) = \tfrac{1}{2} I$$