Taipei Conference

in honor of **Professor Pjek-Hwee Lee**

AN ALGEBRAIC APPROACH TO TROPICAL MATHEMATICS

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§ **1.** *Brief introduction to tropical geometry*

§§ **1.** *Amoebas and their degeneration*

For any complex affine variety $W = \{(z_1, \ldots, z_n) : z_i \in \mathbb{C}\} \subset \mathbb{C}^{(n)}$,

and any small t, *define its* **amoeba** $\mathcal{A}(W)$ *defined as*

$$\{(\log_t |z_1|, \ldots, \log_t |z_n|) : (z_1, \ldots, z_n) \in W\}$$

$$\subset (\mathbb{R} \cup \{-\infty\})^{(n)},$$

graphed according to the (rescaled) coordinates $\log_t |z_1|, \ldots, \log_t |z_n|$.

Note that

$$\log_t |z_1 z_2| = \log_t |z_1| + \log_t |z_2|.$$

Also, if $z_2 = cz_1$ *for* $c << t$ *then*

$$\log_t(|z_1| + |z_2|) = \log_t((|c| + 1)|z_1|) \approx \log_t |z_1|$$

for large enough $|z_1|$*, as* $t \to \infty$*.*

The degeneration $t \to \infty$ *is called the* **tropicalization** *of* W*.*

A more algebraic way of viewing this degeneration of addition:

Define new addition $+_t$ *on* $\mathbb{R}$, *by*

$$a +_t b = \sqrt[t]{a^t + b^t};$$

For $t = 1$ *this is the usual addition, and as* $t \mapsto \infty$ *this becomes the maximum.*

Many invariants (dimension, intersection numbers, genus, etc.) are preserved under tropicalization and become easier to compute by passing to the tropical setting. This tropicalization draws heavily on mathematical analysis, including properties of logarithms. In order to bring in more algebraic techniques, and also permit generic methods, one brings in some valuation theory, following Berkovich and others.

§§ **2.** *A generic passage from (classical) affine algebraic geometry*

Define the Puiseux series *of the form*

$$p(t) = \sum_{\tau \in \mathbb{R}_{\geq 0}} c_\tau t^\tau,$$

where $c_\tau \in \mathbb{C}$ *(or any algebraically closed field of characteristic 0) and the powers of the indeterminate* t *are taken over well-ordered subsets of* $\mathbb{R}$*. For* $p(t) \neq 0$*, define*

$$v(p(t)) \; := \min\{\tau \in \mathbb{R}_{\geq 0} \; : \; c_\tau \neq 0\}.$$

As $t \to 0$*, the dominant term is* $c_{v(p(t))} t^{v(p(t))}$.

The field of Puiseux series $\mathbb{K}$ is algebraically closed, whereas v is a valuation, with respect to which $\mathbb{K}$ is Henselian. On the other hand, Puiseux series serve as generic coefficients of polynomials describing affine varieties, so techniques of valuation theory become applicable.

We replace v by $-v$ to switch minimum to maximum.

§§ **3.** *The max-plus algebra*

A **semiring**[†] $(R, +, \cdot, 1)$ *(without a zero element) is a set* R *equipped with two binary operations* $+$ *and* $\cdot$ *, such that:*

- $(R, +)$ *is an Abelian semigroup;*
- $(R, \cdot\ , 1_R)$ *is a monoid with identity element* 1_R*;*
- *Multiplication distributes over addition.*

A semiring is a semiring† equipped with a zero element 0_R satisfying

$$a + 0_R = a, \quad a \cdot 0_R = 0_R = 0_R \cdot a, \quad \forall a \in R.$$

(A semiring with negatives is a ring.)

The max-plus algebra is a semiring†. It becomes a semiring when one adjoins the zero element $-\infty$. We often delete the zero element, which can be a nuisance.

FOUR NOTATIONS:

Max-plus algebra: $(\mathbb{R}, +, \max, -\infty, 0)$

Tropical notation: $(\mathbb{T}, \odot, \oplus, -\infty, 0)$

Logarithmic notation (for our examples): $(\mathbb{T}, \cdot, +, -\infty, 0)$

Algebraic semiring notation (preferred here): $(R, \cdot, +, 0, 1)$

The launching board of the supertropical algebra is to view $(\mathbb{R}, +)$ *as an ordered Abelian group. Any ordered monoid* $\mathcal{M}$ *gives rise to a semiring*†*, where multiplication is the monoid operation, and addition is taken to be the maximum. (Usually* $\mathcal{M}$ *is taken to be a group.) This semiring is* **bipotent** *in the sense that* $a+b \in \{a, b\}$. *Thus, the max-plus (tropical) algebra is viewed algebraically as a bipotent semiring*†*.*

Conversely, any bipotent semiring† *becomes an ordered monoid, taking* $a \leq b$ *if and only if* $a+b = b$.

§§ **4.** *Polynomials and functions*

For any semiring† R, *one can define the semiring*† $R[\lambda]$ *of* **polynomials**, *where polynomial addition and multiplication are defined in the familiar way:*

$$\left(\sum_i \alpha_i \lambda^i\right)\left(\sum_j \beta_j \lambda^j\right) = \sum_k \left(\sum_{i+j=k} \alpha_i \beta_{k-j}\right)\lambda^k.$$

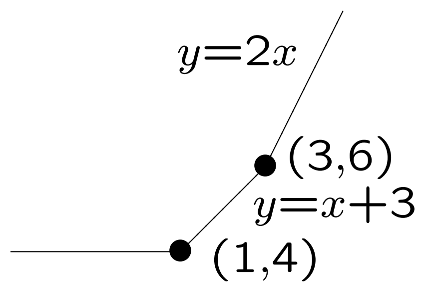
Likewise, one defines polynomials $F[\Lambda]$ *in indeterminates* Λ*.*

Bipotence fails for polynomials: $(2\lambda + 1) + (\lambda + 2) = 2\lambda + 2.$

As usual, any function is described in terms of its graph. The graph of a polynomial in one indeterminate over the max-plus algebra is piecewise linear; over several indeterminates we get a polytope, closely related to the Newton polytope.

In contrast to the classical theory of algebras over an infinite field, different polynomials over the max-plus algebra may have the same graph, i.e, behave as the same function. For example, $\lambda^2 + \lambda + 7$ *and* $\lambda^2 + 7$ *are the same over the max-plus algebra.*

Graph of $\lambda^2 + 3\lambda + 4$:



Its graph (rewritten in standard notation) consists of the horizontal line $y = 1$ up to $x = 1$, at which point it switches to the line segment $y = x + 3$ until $x = 3$, and then to the line $y = 2x$.

§§ **5.** *The function semiring*†

Given a set S *and semiring*† R*, define* $\mathsf{Fun}(S, R)$ *to be the set of functions from* S *to* R*, which becomes a semiring*† *under componentwise operations.*

For any subset $S \subseteq R^{(n)}$*, there is a natural homomorphism*

$$\Phi : R[\lambda_1, \dots, \lambda_n] \to \mathsf{Fun}(S, R),$$

and we view each polynomial in terms of its image in $\mathsf{Fun}(S, R)$*.*

§§ **6.** *Corner loci in tropicalizations*

Suppose

$$f = \sum_{\mathbf{i}\in\mathbb{N}^{(n)}} p_{\mathbf{i}}(t)\lambda_1^{i_1}\cdots\lambda_n^{i_n},$$

where $p_{\mathbf{i}}(t) \in \mathbb{K}$, *and let* $v : \mathbb{K} \to \mathbb{R}$ *be as above. Define its* **tropicalization**

$$\tilde{v}(f) = \sum_{\mathbf{i}\in\mathbb{N}^{(n)}} v(p_{\mathbf{i}}(t))\lambda_1^{i_1}\cdots\lambda_n^{i_n}.$$

Basic fact: If $\sum a_i = 0$, *then* $v(a_{i_1}) = v(a_{i_2})$ *for some* $i_1 \neq i_2$.

The image under $\tilde{v}$ of any root of f (over the max-plus algebra) must be a point on which the maximal evaluation of f on its monomials is attained by at least two monomials. This is called a **corner root**, *and the set of corner roots is called the* **corner locus**. *This brings us back to the max-plus algebra.*

The corner locus is the domain of non-differentiability of the graph of f.

Example 1. *The polynomial* $\lambda^2 + 3\lambda + 4$ *over the max-plus algebra has corner locus* $\{1, 3\}$ *since*

$$0 \cdot 3^2 = 3 \cdot 3 = 6, \qquad 3 \cdot 1 = 4.$$

§§ **7.** *Kapranov's Theorem*

Theorem 1 (Kapranov)**.** *The tropicalization of the zero set of a polynomial f coincides with the corner locus of the tropicalization of f.*

Example 2. $f = 10t^2\lambda^3 + 9t^8$ *has the root* $\lambda \mapsto a = -\sqrt[3]{\frac{9}{10}}t^2$. *Then*

$$\tilde{v}(f) = 2\lambda^3 + 8$$

has the corner root $v(a) = 2$.

If instead

$$f = (8t^5 + 10t^2)\lambda^3 + (3t+6)\lambda^2 + (7t^{11} + 9t^8),$$

then again $\tilde{v}(f) = 2\lambda^3 + 0\lambda^2 + 8$, *which as a function equals* $2\lambda^3 + 8$ *and again has the corner root 2.*

One can lift this corner root to a root of f *by building up a Puiseux series with lowest term* $-\sqrt[3]{\frac{9}{10}}t^2$, *using valuation-theoretic methods.*

§§ **8.** *Nice properties of bipotent semiring†s*

- *Any bipotent semiring† satisfies the* **Frobenius property***:*

$$\Big(\sum a_i\Big)^m = \sum a_i^m \tag{1}$$

for **every** *natural number* m*.*

- *Any polynomial in one indeterminate can be factored by inspection, according to its corner locus.*

  For example, $\lambda^4 + 4\lambda^3 + 6\lambda^2 + 5\lambda + 3$ *has corner locus*

$$\{-2, -1, 2, 4\}$$

  and factors as

$$(\lambda + 4)(\lambda + 2)(\lambda + (-1))(\lambda + (-2)).$$

§§ **9.** *Poor properties of bipotent semiring†s*

Unfortunately, bipotent semiring†s have two significant drawbacks:

- *Bipotence does not reflect the true nature of a valuation v. If $v(a) \neq v(b)$ then $v(a+b) \in \{v(a), v(b)\}$, but if $v(a) = v(b)$ we do not know much about $v(a+b)$. For example, the lowest terms in two Puiseux series may or may not cancel when we take their sum.*

- *Distinct cosets of ideals need not be disjoint. For any ideal I, given $a, b \in R$, if we take $c \in I$ large enough, then*

$$a + c = c = b + c \in (a + I) \cap (b + I).$$

This complicates homomorphisms and factor structures. One does not describe semiring homomorphisms via kernels, but rather via congruences, which is much more complicated.

As a consequence, the literature concerning the structure of max-plus semiring†s is limited. There are remarkable theorems, but they are largely combinatoric in nature, and often the statements are hampered by the lack of a proper language.

In summary, the max-plus algebra is too coarse a degeneration for an full study of the algebraic theory of tropical mathematics. The objective of our research is to describe a less severe structure that provides the language (and basic results) for a useful structure theory.

§§ **10.** *Supertropical domains*[†]

We modify the max-plus structure on a given ordered Abelian monoid $\mathcal{M}$*, which we denote as* R_∞*, by considering a cover of* R_∞*.*

Take a monoid surjection $\nu : R_1 \to R_\infty$*. Often* ν *is an isomorphism. We write* a^ν *for* $\nu(a)$*, for each* $a \in R_1$*.*

The disjoint union $R := R_1 \cup R_\infty$ *becomes a multiplicative monoid under the given monoid operations on* R_1 *and* R_∞*, when we define* ab^ν, $a^\nu b$ *both to be* $a^\nu b^\nu \in R_\infty$.

We extend ν *to the* **ghost map** $\nu : R \to R_\infty$ *by taking* ν *to be the identity on* R_∞*. Thus,* ν *is a monoid projection.*

We make R *into a semiring*† *by defining*

$$a + b = \begin{cases} a \text{ for } a^\nu > b^\nu; \\ b \text{ for } a^\nu < b^\nu; \\ a^\nu \text{ for } a^\nu = b^\nu. \end{cases}$$

R so defined is called a **supertropical domain**†.

Special Case: A **supertropical 1-semifield**† *is a supertropical domain*† *for which R_1 is an Abelian group.*

Another way to view multiplication is to define $\nu_i : R_1 \to R_i$ (for $i \in \{1, \infty\}$ by $\nu_1 = 1_{R_1}$ and $\nu_\infty = \nu$. Then multiplication is given by

$$\nu_i(a)\nu_j(b) = \nu_{ij}(ab).$$

R_∞ is a semiring† ideal of R. R_1 is called the **tangible submonoid** *of R. R_∞ is called the* **ghost ideal**, *also denoted as $\mathcal{G}$.*

The motivation: The ghost ideal $\mathbb{R}_\infty$ is to be treated much the same way that one would customary treat the 0 element in commutative algebra. (Also, we can formally adjoin a zero element in an extra component R_0.)

Examples.

- $R_1 = (\mathbb{R}, +)$, $R_\infty = (\mathbb{R}, +)$, *and* ν *is the identity map (Izhakian's original example);*

- $R_1 = F^\times$ *(*F *a field),* R_∞ *an ordered group, and* $\nu : F^\times \to R_\infty$ *is a valuation. Note that we forget the original addition on the field* F!

Towards this end, we write

$$a \underset{\mathcal{G}}{\models} b \quad \text{if} \quad a = b \text{ or } a = b + \text{ghost}.$$

(Accordingly, write $a \underset{\mathcal{G}}{\models} 0$ *if* a *is a ghost.)*

Note that for a *tangible,* $a \underset{\mathcal{G}}{\models} b$ *iff* $a = b$.

This partial order $\underset{\mathcal{G}}{\models}$*, called* **ghost surpasses***, is of fundamental importance in the supertropical theory, replacing equality in many analogs of theorems from commutative algebra.*

Supertropical domains† *also satisfy the Frobenius property given above in* (1). *A suggestive interpretation: For any m there is a semiring*† *endomorphism $R \to R$ given by $f \mapsto f^m$, reminiscent of the Frobenius automorphism in classical algebra. But here the Frobenius property holds for every m. This plays an important role in our theory, and is called* **characteristic 1** *in the literature.*

§§ **11.** *Ghost roots*

If a polynomial $f \neq 0$, then f cannot have any zeroes in the classical sense! But here is an alternate definition.

An n-tuple $\mathbf{a} = (a_1, \dots, a_n) \in R^{(n)}$ is called a (ghost) **root** *of a polynomial $f \in R[\lambda_1, \dots, \lambda_n]$ if $f(\mathbf{a}) \underset{\mathcal{G}}{\models} 0$, i.e., if $f(\mathbf{a}) \in \mathcal{G}$.*

There are two kinds of roots of a polynomial $f = \sum h_{\mathbf{j}}$*:*

Case I *At least two of the* $h_{\mathbf{j}}(\mathbf{a})^{\nu}$ *are maximal (and thus equal), in this case*

$$f(\mathbf{a}) = h_{\mathbf{j}}(\mathbf{a})^{\nu} \in \mathcal{G}.$$

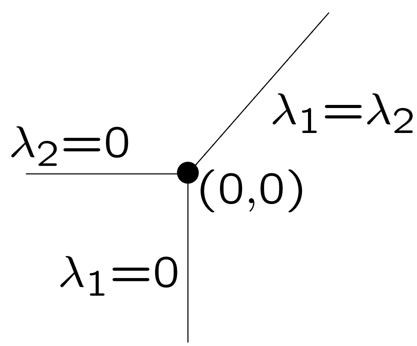
Tangible roots in this case are just the **corner roots***.*

Case II *There is a unique* $\mathbf{j}$ *for which* $h_{\mathbf{j}}(\mathbf{a})^{\nu}$ *is maximal. Then* $f(\mathbf{a}) = h_{\mathbf{j}}(\mathbf{a})$ *is ghost when the coefficient of* $h_{\mathbf{j}}$ *is ghost.*

Example 3. *(The tropical line) The tangible roots in* $D(\mathbb{R})[\lambda]$ *of the polynomial* $f = \lambda_1 + \lambda_2 + 0$ *are:*

$$\begin{cases} (0,a) \text{ for } a \leq 0; \\ (a,0) \text{ for } a \leq 0; \\ (a,a) \text{ for } a \geq 0. \end{cases}$$

The "curve" of tangible roots of f is comprised of three rays, all emanating from $(0, 0)$.



§§ **12.** *The tropical version of the algebraic closure*

A semiring† R *is* **divisibly closed** *if* $\sqrt[m]{a} \in R$ *for each* $a \in R$. *There is a standard construction to embed a 1-semifield*† *into a divisibly closed, supertropical 1-semifield*†.

Example: The divisible closure of the max-plus semifield† $\mathbb{Z}$ *is* $\mathbb{Q}$, *which is closed under taking roots of polynomials.*

Theorem 1. *If two polynomials agree on an extension of a divisibly closed supertropical 1-semifield*† R*, then they already agree on* R*.*

The proof is an application of Farkas' theorem from linear inequalities! (It can also be seen via general principles of model completeness.)

§§ **13.** *Factorization*

An example of an irreducible quadratic polynomial:

$$\lambda^2 + 5^\nu \lambda + 7.$$

(although $\lambda^2 + 5\lambda + 7 = (\lambda + 2)(\lambda + 5)$*). But this is the only kind of example:*

Theorem 2. *Any polynomial over a divisibly closed 1-semifield*† *is the product (as a function) of linear polynomials and quadratic polynomials of the form* $\lambda^2 + a^\nu \lambda + b$*, where* $\frac{b}{a} < a$*.*

Unique factorization can fail, even with respect to equivalence as functions.

In one indeterminate:

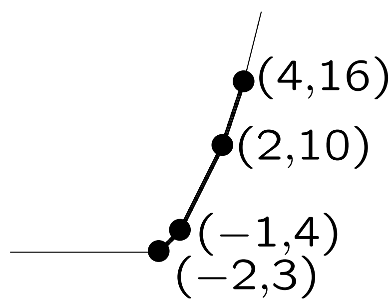
$$\lambda^4 + 4^\nu\lambda^3 + 6^\nu\lambda^2 + 5^\nu\lambda + 3$$

$$= (\lambda^2 + 4^\nu\lambda + 2)(\lambda^2 + 2^\nu\lambda + 1)$$

$$= (\lambda^2 + 4^\nu\lambda + 2)(\lambda + (-1))(\lambda + 2)$$

$$= (\lambda^2 + 4^\nu\lambda + 3)(\lambda^2 + 2^\nu\lambda + 0).$$

Graph of $\lambda^4 + 4^\nu\lambda^3 + 6^\nu\lambda^2 + 5^\nu\lambda + 3$**:**



(The thickened line segments in the middle indicate that all of these segments are roots.)

A geometrical interpretation of these factorizations: The tangible root set of f is the interval $[-2, 4]$, where -1 and 2 also are corner roots.

The tangible root set of an irreducible quadratic factor $\lambda^2 + a^\nu \lambda + b$ is the closed interval $[\frac{b}{a}, a]$, and the union of these segments must correspond to the root set of f. Different decompositions of the tangible root set yield different factorizations.

The decomposition for the factorization

$(\lambda^2 + 4^{\nu}\lambda + 2)(\lambda + (-1))(\lambda + 2)$ *is*

$$[-2, 4] \cup \{-1\} \cup \{2\},$$

which best matches the geometric intuition. The decompositions for the other factorizations:

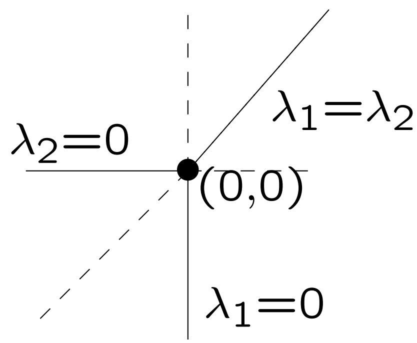
$$[-2, 4] \cup [-1, 2]; \qquad [-1, 4] \cup [-2, 2].$$

In two indeterminates, we have a worse situation:

$$
\begin{aligned}
&(0+\lambda_1+\lambda_2)(\lambda_1+\lambda_2+\lambda_1\lambda_2) \\
&\quad = \lambda_1+\lambda_2+\lambda_1^2+\lambda_2^2 \\
&\qquad + \nu(\lambda_1\lambda_2)+\lambda_1^2\lambda_2+\lambda_2^2\lambda_1 \\
&\quad = (0+\lambda_1)(0+\lambda_2)(\lambda_1+\lambda_2).
\end{aligned}
$$

Geometrical interpretation: A tropical variety can decompose in different ways as the union of irreducible varieties (here, either as a tropical line together with a tropical conic, or three rays):

Decompositions of $\lambda_1^2 + \lambda_2^2 + \lambda_1^2\lambda_2 + \lambda_2^2\lambda_1 + \lambda_1 + \lambda_2$**:**



§ **2.** *Supertropical matrix theory*

Assume $R = (R, \mathcal{G}, \nu)$ *is a commutative supertropical domain*†*.* *One defines the* **matrix semiring**† $M_n(R)$ *in the usual way.*

Since -1 *is not available in tropical mathematics, our main tool in linear algebra is the permanent* $|A|$*, which can be defined for any matrix* A *over* R *as*

$$\left|(a_{i,j})\right| = \sum_{\pi \in S_n} a_{\pi(1),1} \cdots a_{\pi(n),n}. \qquad (2)$$

.

This notion is not very useful over the usual max-plus semiring†*:*

Example 4. $A = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$ *(over the max-plus semiring*† $\mathbb{Z}$*).* $|A| = 2$, *but* $A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, *so*

$$\left|A^2\right| = 5 \neq 4 = |A|^2 .$$

Although the permanent is not multiplicative in general, it is in the supertropical theory, in a certain sense, and enables us to formulate many basic notions from classical matrix theory. Thus, the permanent is also called the **supertropical determinant**.

Definition 1. A matrix A is **nonsingular** if $|A|$ is tangible; A is **singular** when $|A| \in \mathcal{G}_0$.

Example 5. *In Example 4, A^2 is singular with $\left|A^2\right| = 5^\nu$.*

Theorem 3. *For any* $n \times n$ *matrices over a supertropical semiring* R*, we have*

$$|AB| \underset{\mathcal{G}}{\models} |A|\,|B|\,.$$

In particular, $|AB| = |A|\ |B|$ *whenever* AB *is nonsingular.*

Definition 2. The **minor** $A'_{i,j}$ is obtained by deleting the i row and j column of A. The **adjoint** matrix $\text{adj}(A)$ is the transpose of the matrix $(a'_{i,j})$, where $a'_{i,j} = \left|A'_{i,j}\right|$.

Theorem 4.

1. $|A \ \mathrm{adj}(A)| = |A|^n$.

2. $|\,\mathrm{adj}(A)| = |A|^{n-1}$.

The proof of equality (rather than just ghost surpasses) follows directly from the celebrated Birkhoff-Von Neumann Theorem, which states that every positive doubly stochastic $n \times n$ *matrix is a convex combination of at most* n^2 *cyclic covers.*

Definition 3. A **quasi-identity** is a multiplicatively idempotent matrix of tropical determinant 1, equal to the identity on the diagonal and ghost off the diagonal.

Theorem 5. *For any nonsingular matrix A over a supertropical 1-semifield† F,*

$$A \,\mathrm{adj}(A) = |A| \, I_A,$$

for a suitable quasi-identity matrix I_A.

Likewise $\mathrm{adj}(A)A = |A|\, I'_A$, *for a suitable quasi-identity matrix* I'_A.

$(I'_{\mathrm{adj}(A)} = I_A.)$

The adjoint also is used to solve the matrix equations $Ax \underset{\mathcal{G}}{\models} v$ *for tangible vectors* x, v.

The supertropical version of the Hamilton-Cayley theorem: The matrix A **satisfies** *a polynomial $f \in R[\lambda]$ if $f(A) \underset{\mathcal{G}}{\models} (0)$; i.e.,*

$$f(A) \in M_n(\mathcal{G}).$$

Theorem 6. *Any matrix A satisfies its characteristic polynomial $f_A = |\lambda I + A|$.*

Example 6. *The characteristic polynomial* f_A *of the matrix*

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

over $F = D(\mathbb{R})$*, is*

$$(\lambda + 4)(\lambda + 1) + 0 = (\lambda + 4)(\lambda + 1),$$

and indeed the vector $(4, 0)$ *is an eigenvector of* A*, with eigenvalue 4. However, there is no eigenvector having eigenvalue 1.*

Definition 4. A vector v is a **supertropical eigenvector** of A, with **supertropical eigenvalue** $\beta \in \mathcal{T}$, if

$$Av \underset{\mathcal{G}}{\models} \beta v.$$

In Example 6, $(0,4)$ is a supertropical eigenvector of A having eigenvalue 1, although it is not an eigenvector.

Theorem 7. *The roots of the polynomial f_A are precisely the supertropical eigenvalues of A.*

§ **3.** *Tropical dependence of vectors*

Definition 5. A subset $W \subset R^{(n)}$ is **tropically dependent** if there is a finite sum $\sum \alpha_i w_i \in \mathcal{G}_0^{(n)}$, with each α_i tangible; otherwise W is called **tropically independent**.

Here is our hardest theorem:

Theorem 8. *Suppose R is a supertropical domain†. The following three numbers are equal for a matrix:*

- *The maximum number of tropically independent rows;*
- *The maximum number of tropically independent columns;*
- *The maximum size of a square nonsingular submatrix of A.*

Surprise: Even when the characteristic polynomial factors into n distinct linear factors, the corresponding n eigenvectors need not be supertropically independent!

Example 7.

$$A = \begin{pmatrix} 10 & 10 & 9 & - \\ 9 & 1 & - & - \\ - & - & - & 9 \\ 9 & - & - & - \end{pmatrix}. \tag{3}$$

The characteristic polynomial of A is

$$f_A = \lambda^4 + 10\lambda^3 + 19\lambda^2 + 27\lambda + 28,$$

whose roots are $10, 9, 8, 1$, *which are the eigenvalues of* A.

The four supertropical eigenvectors comprise the matrix

$$V = \begin{pmatrix} 30 & 28 & 25 & 12 \\ 29 & 28 & 26 & 27 \\ 28 & 28 & 27 & 28 \\ 29 & 28 & 26 & 20 \end{pmatrix},$$

which is singular, having determinant 112^{ν}.

This difficulty is resolved by passing to asymptotics, i.e., high enough powers of A. *In contrast to the classical case, a power of a nonsingular* $n \times n$ *matrix can be singular (and even ghost).*

§ **4.** *The resultant*

We now have all the tools at our disposal to define the supertropical resultant, in terms of Sylvester matrices, which enables us to determine when two polynomials have a common root and yields an algebraic proof of a version of Bezout's theorem.

§ **5.** *Layered structure*

Although the supertropical domain† *is successful in tropical linear algebra, it is still too coarse for many applications in geometry and calculus.*

In order to handle multiple roots and derivatives, we need to consider multiple ghost layers, "sorted" over an ordered semiring L *whose elements are all presumed to be positive or 0.*

Our main objectives with the layered structure:

- *Extend the scope of the supertropical theory, as well as the max-plus theory. For example, we can treat multiple roots by means of layers.*

- *Obtain proofs which are actually more natural in this context than in the more special supertropical theory.*

- *Relate various concepts to notions in the tropical literature.*

The basic construction:

Given a semiring† $\mathcal{M}$, *for any semiring*† L *define* $R := R(L, \mathcal{M})$ *to be set-theoretically* $L \times \mathcal{M}$, *where* (ℓ, a) *is denoted as* ${}^{[\ell]}a$.

R *is a semiring*† *with multiplication given by:*

$$ {}^{[k]}a \ {}^{[\ell]}b = {}^{[k\ell]}(ab)\,, \tag{4}$$

and addition by:

$$ {}^{[k]}a + {}^{[\ell]}b = \begin{cases} {}^{[k]}a & \text{if } a > b, \\ {}^{[\ell]}b & \text{if } a < b, \\ {}^{[k+\ell]}a & \text{if } a = b. \end{cases} \tag{5}$$

We define the **sort map** $s : R \to L$ *by taking* $s({}^{[k]}a) = k$, *and the* k-**layer** *of* R *is the pre-image of* k *in* R.

The familiar max-plus algebra is recovered by taking $L = \{1\}$*, whereas the standard supertropical structure is obtained when* $L = \{1, \infty\}$*.*

In general, the 1-layer is a multiplicative monoid corresponding to the tangible elements in the standard supertropical theory, and the ℓ*-layers for* $\ell > 1$ *correspond to the ghosts in the standard supertropical theory.*

*Unique factorization fails in the standard supertropical theory. Taking $L = \mathbb{N}$ yields enough refinement to permit us to utilize some tools of mathematical analysis. Taking $L = \mathbb{Q}_{>0}$ "almost" restores unique factorization in one indeterminate. The sticky issue here is polynomials with a single root a, which we call a-***primary***; these have the form*

$$\sum_i a^i \lambda^{d-i}.$$

Any polynomial in one indeterminate can be factored uniquely into its primary factors.

(Unique factorization in several indeterminates still fails in certain situations, but for the geometric reason that certain tropical hypersurfaces can be decomposed non-uniquely).

We can layer Fun(S, R) *with respect to* Fun(S, L) *by means of the sort function*

$$s : \mathsf{Fun}(S, R) \to \mathsf{Fun}(S, L)$$

defined as follows: For $f \in$ Fun(S, R), *we take* $s(f) \in$ Fun(S, L) *via*

$$s(f) : \mathbf{a} \mapsto s(f(\mathbf{a})).$$

§§ **14.** *Layered varieties*

Here are the main geometric definitions.

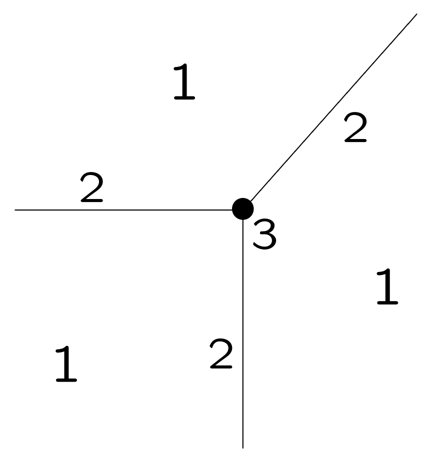
Definition 6. An element $\mathbf{a} \in \mathcal{S}$ is a **corner root** of a polynomial f if $f(\mathbf{a}) \neq h(\mathbf{a})$ for each monomial h of f.

(In other words, we need at least two monomials to attain the appropriate ghost level of $f(\mathbf{a})$.)

The **corner locus** $Z_{\mathsf{corn}}(\mathcal{I})$ *of* $\mathcal{I} \subset R[\Lambda]$ *is the set of simultaneous corner roots of the functions in* $\mathcal{I}$*. Any such corner locus will also be called an (affine)* **layered variety**.

The (affine) **coordinate semiring**† *of a layered variety* Z *is* $R[\Lambda] \cap \mathrm{Fun}(Z, R)$.

The layering for the tropical line $f(x) = x + y + 0$.



Different algebraic definitions of dimension of the coordinate semiring† (chains of prime congruences, Gelfand-Kirillov dimension, module-theoretic Krull dimension in the spirit of Gordon and Robson) all give rise to a natural notion of dimension of the corresponding layered variety. We conjecture that these are all the same, over a layered domain†.

§§ **15.** *The component topology*

Definition 7. Write $f = \sum_{\mathbf{i}} f_{\mathbf{i}}$ for $\mathbf{i} = (i_1, \ldots, i_n)$, a sum of monomials.

Define the **components** $D_{f,\mathbf{i}}$ of f to be

$$D_{f,\mathbf{i}} := \{\mathbf{a} \in \mathcal{S} : f(\mathbf{a}) = f_{\mathbf{i}}(\mathbf{a})\}.$$

Suppose $f = \sum_{\mathbf{i}} f_{\mathbf{i}}$ *and* $g = \sum_{\mathbf{j}} g_{\mathbf{j}}$*. Then*

$$D_{f,\mathbf{i}} \cap D_{g,\mathbf{j}} = D_{fg,\mathbf{i}+\mathbf{j}}.$$

Hence, the set of components of polynomials comprises a base for a topology on S.

There is a **layered Nullstellensatz** *in this context.*

Thank you for your attention.