Some Recent Developments in the Theory of Baer and Rickart Modules

(joint work with C. Roman and G. Lee)

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(ICRA, Taipei, Taiwan- In honor of Prof. P.-H. Lee, July 11)

Definition

A ring R is called a Baer ring
if the right annihilator of every subset in R is generated by an idempotent, i.e. $r_R(K) = \{x | Kx = 0\} = eR \ \exists e^2 = e$ for any nonempty subset $K \subseteq R$.
$\Leftrightarrow$ the left annihilator of every subset in R is generated by an idempotent, i.e. $l_R(K) = \{x | xK = 0\} = Re \ \exists e^2 = e$ for any nonempty $K \subseteq R$ (i.e. the notion is left-right symmetric)

Example

- Roots in Functional Analysis- the Algebra of bounded operators on a Hilbert space, in fact any von Neumann Algebra is a Baer (*-)ring;
- any domain;
- any right noetherian hereditary ring is a Baer ring;
- any von Neumann regular ring + lattice of principal rt ideals complete (e.g., any right selfinjective regular ring; in fact any right nonsingular extending ring) is Baer;
-the Boolean ring of all subsets of a given set

Definition

A ring R is called a right Rickart ring (also known as a right PP ring) if the right annihilator of any element of R is generated by an idempotent as a right ideal, i.e., $\forall a \in R$, $r_R(a) = eR$, $\exists e^2 = e \in R$.

Example

- von Neumann regular rings;
- Baer rings, (For example, every right nonsingular (hence regular) right self injective ring);
- right (semi-)hereditary rings;
- $End_R(R^{(\mathcal{I})})$ with R a right hereditary ring and $\mathcal{I}$ an index set, is a right Rickart ring. - A right hereditary ring which is not a Baer ring is right Rickart but not Baer.

* left Rickart (p.p.) rings are defined similarly.
* There is a well-known example of Chase of a right Rickart ring which is not left Rickart.(i.e. the notion is NOT left-right symmetric)

* In the general module theoretic setting, the concept of Baer modules using the endomorphism ring of the module, was introduced in 2004.
* Throughout this talk, let M be a right R-module and $S = End(M)$.(We will see that this notion heavily depends on S)

Definition (T. Rizvi and C. Roman, 2004)

A right R-module M is called a Baer module if $\forall\ N \leq M, l_S(N) = Se$ with $e^2 = e \in S$.
$\Leftrightarrow$if $\forall\ {}_SI \leq S, r_M(I) = eM$ with $e^2 = e \in S$

Recall that the annihilators are given by $l_S(N) = \{\phi \in S | \phi N = 0\}$ and $r_M(I) = \{m \in M | Im = 0\}$.
Examples: (i) If R is a Baer ring and $e^2 = e \in R$ then eR is a Baer R-module, (as is R_R.) (ii) Any free module of countable rank over a PID R, is a Baer R-module. In particular, $\mathbb{Z}^n$ is a Baer $\mathbb{Z}$-module, $\forall\ n \in \mathbb{N}$. (iii) All semisimple modules are obviously Baer modules. (iv) Every nonsingular injective (even extending) module is a Baer module. (Recall that a module M is called extending if every submodule N of M is essential (large) in a direct summand of M i.e. $N \leq^e P$ where P is a direct summand of M.)

Rickart Modules

* To obtain a module theoretic analogue for right Rickart rings via endomorphism rings we restrict the Baer module definition to single elements. One motivation for our study is the question: If R is a right Rickart ring then what can be said about the right R module eR?

Definition

Let M be a right R-module and let $S = End_R(M)$.
Then M is called a Rickart module if the right annihilator in M of any single element of S is generated by an idempotent in S.
Equivalently, $\forall \varphi \in S$, $r_M(\varphi) = Ker\varphi = eM$ for some $e^2 = e \in S$.

Note that $Ker\varphi = r_M(\varphi) = r_M(S\varphi)$ for $\varphi \in S$.

Examples

- R_R is a Rickart module if R is a Baer ring, a von Neumann regular ring, or a right hereditary ring; right Rickart ring
- any semisimple module; any Baer module (e.g., every n.s. CS);
- The free $\mathbb{Z}$-module $\mathbb{Z}^{(\mathcal{I})}$, for any index set $\phi \neq \mathcal{I}$,(while $\mathbb{Z}^{(\mathcal{I})}$ is not a Baer $\mathbb{Z}$-module if $\mathcal{I}$ is uncountable). In particular, $\mathbb{Z}^{(\mathbb{R})}$ is a Rickart $\mathbb{Z}$-module but is not a Baer $\mathbb{Z}$-module.
- every proj. module over a rt. hered. ring, is Rickart but may not be Baer module.

Known: Every Baer/Rickart ring is nonsingular. Similarly, every Baer/Rickart module satisfies a certain type of nonsingularity which we call $\mathcal{K}$-nonsingularity.

Definition

We say a module M $\mathcal{K}$-nonsingular iff $Ker\varphi = r_M(\varphi) \leq^e M$ implies $\varphi = 0$, for all $\varphi \in End(M)$.

Proposition

M is a nonsingular module $\Rightarrow$ M is polyform $\Rightarrow$ M is $\mathcal{K}$-nonsingular.

(A module M is called non-M-singular or polyform if $\forall\ N \leq M$, $0 \neq \varphi : N \to M$, $Ker\varphi$ is not esential in N.)
-The reverse implications are not true. The $\mathbb{Z}$-module $\mathbb{Z}_p$ ($p \in \mathbb{Z}$ any prime), is a $\mathcal{K}$-nonsingular module which is not nonsingular.
-For the case of a ring R, the notions coincide.

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Baer and Rickart Rings
Module Theoretic Versions of Baer and Rickart Rings
Connections to the Endomorphism Ring
The Direct Sum Problem
Free Baer Modules and Classes of Rings
Free Rickart Modules and Classes of Rings
IMPORTANT!

* Some properties of a Baer module:

Theorem

(i) *Every direct summand of a Baer module is a Baer module.*

(ii) *Every Baer module is $\mathcal{K}$-nonsingular.*

(iii) *Every Baer M module satisfies the SSIP (the intersection of any family of direct summands of M is a direct summand).*

(iv) *A finitely generated $\mathbb{Z}$-module M is Baer if and only if M is semisimple or torsion-free.*

* Some properties of a Rickart module:

Theorem

(i) *Every direct summand of a Rickart module is a Rickart module.*

(ii) *Every Rickart module is $\mathcal{K}$-nonsingular.*

(iii) *Every Rickart module has the SIP (the intersection of any two direct summands is a direct summand).*

Corollary

If R is a (Baer) right Rickart ring then $M = eR$, $e^2 = e$, is a (Baer) Rickart R-module.

This answers a question posed earlier.

A connection between Baer and Rickart Modules:

Theorem

A module M is Baer if and only if M has the strong summand intersection property and $Ker(\varphi) \leq^{\oplus} M, \forall\ \varphi \in S$ (i.e. M is Baer iff M satisfies the SSIP and M is Rickart).

A well-known result of L. Small can be extended to a general module theoretic setting as follows. Recall:

Theorem

(Small) Let R be a ring that has no infinite set of orthogonal nonzero idempotents. Then the following conditions are equivalent:

(a) *R is a right Rickart ring;*

(b) *R is a Baer ring.*

Using the module theoretic methods we prove:

Theorem

Let M be a module and let $S = End_R(M)$ have no infinite set of orthogonal nonzero idempotents. Then the following conditions are equivalent:

(a) *M is a Rickart module;*

(b) *M is a Baer module.*

There exist close links between Baer modules and extending modules via $\mathcal{K}$-nonsingularity.
First recall **Chatters-Khuri Theorem**: A ring R is a right nonsingular right extending ring iff R is Baer and right co-nonsingular ring. (R is right cononsingular if rt. annihilator of a nonessential rt. ideal is nonzero.)

Proposition

Every Baer module is $\mathcal{K}$-nonsingular.

Proposition

$\mathcal{K}$-nonsingular extending modules are Baer.

(This provides a rich source of examples of Baer modules, e.g. any nonsingular injective or extending module is Baer. In general $M/Z_2(M)$ is a Baer module for any extending module M, where $Z_2(M)$ is the second singular submodule of M)
Now an important result:

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Now an important result:

Theorem (R-R, 2004)

A module is $\mathcal{K}$-nonsingular, extending if and only if it is Baer and $\mathcal{K}$-cononsingular.

(A module M is called $\mathcal{K}$-cononsingular if $\forall\ N \leq M$, $\varphi N \neq 0$ for all $0 \neq \varphi \in S$ implies $N \leq^e M$.)

Connections of Baer and Rickart modules to their Endomorphism Rings

Theorem

Let M be a Baer (respectively, Rickart) module. Then $S = End(M)$ is a Baer (respectively, right Rickart) ring.

Converse not true:

Example

Let $M = \mathbb{Z}_{p^\infty}$, considered as a right $\mathbb{Z}$-module. Then it is well-known that $S = End_{\mathbb{Z}}(M)$ is the ring of p-adic integers, a commutative domain. Thus it is a Baer (and hence Rickart) ring. However $M = \mathbb{Z}_{p^\infty}$ is not (even a Rickart) a Baer module.

*A characterization for the case of Baer Modules

Theorem (Rizvi-Roman, J. Algebra 2009)

A module M is a Baer module if and only if its endomorphism ring S is a Baer ring and M is quasi-retractable.

(A module M is called *quasi-retractable* if $Hom(M, r_M(I)) \neq 0$, $\forall$ $0 \neq r_M(I)$, $I \leq {}_S S$ (or, equivalently, if $r_M(I) \neq 0$ then $r_S(I) \neq 0$, $\forall$ $I \leq {}_S S$)).

Corollary

R^n is a Baer module iff $M_n(R)$ is a Baer ring ($n \in \mathbb{N}$).

*A characterization for the case of Rickart Modules

Theorem (Lee-Rizvi-Roman, Comm. Algebra 2011)

The following conditions are equivalent:

(a) *M is a Rickart module;*

(b) *$S = End(M)$ is a right Rickart ring and M is k-local-retractable.*

(M is called k-local-retractable if for any $\varphi \in S = End(M)$ and any nonzero element $m \in r_M(\varphi)$ in M, there exists a nonzero homomorphism $\psi_m \in S$ such that $m \in \psi_m(M) \subseteq r_M(\varphi) = Ker\varphi$)

Corollary

The free R-module $R^{(n)}$ is Rickart if and only if $\mathbb{M}_n(R)$ is a right Rickart ring for any $n \in \mathbb{N}$.

***Direct sums of Baer (resp., Rickart) Modules are not always Baer (resp., Rickart)**

We have seen earlier that any direct summand of a Baer or Rickart module inherits the property however a direct sum of Baer (resp.,Rickart) modules does not:

Example

$\mathbb{Z}$ and $\mathbb{Z}_2$ are Baer $\mathbb{Z}$-modules ($\mathbb{Z}$ is a domain; $\mathbb{Z}_2$ is simple). By However $\mathbb{Z} \oplus \mathbb{Z}_2$ is not a Baer (or even a Rickart) module (For $\varphi(n, \hat{m}) = \hat{n}$ the $Ker(\varphi) = 2\mathbb{Z} \oplus \mathbb{Z}_2$, is not a direct summand since $2\mathbb{Z} \oplus \mathbb{Z}_2 \neq \mathbb{Z} \oplus \mathbb{Z}_2$, as $\mathbb{Z}$ is uniform).

Proposition

If M is an indecomposable Rickart module which has a nonzero maximal submodule N, then $M \oplus (M/N)$ is not a Rickart module, while M and M/N are Baer modules.

***When are Direct Sums of Baer (resp., Rickart) Modules, Baer (resp., Rickart)?**

Finding necessary and sufficient conditions for direct sums of extending modules to be extending, has been an open question for well over 2 decades. A sufficient condition for a finite direct sum of extending modules to be extending is that each summand be relatively injective to all others (Harmanci-Smith). We show that an analogue holds true also for the case of Baer modules.

Theorem

Let $\{M_i\}_{1\leq i\leq n}$ be a class of Baer (resp., Rickart) modules, where $n \in \mathbb{N}$. Assume that M_i is M_j-injective for all $i < j$. Then $\bigoplus_{i=1}^{n} M_i$ is a Baer (resp., Rickart) module iff M_i is M_j-Rickart.

(We call a module M_i relatively Rickart to N (or N-Rickart) if, for every homomorphism $\varphi : M_i \to N$, $Ker\varphi \leq^{\oplus} M_i$.)

Recall that a module M is called relatively C_2 to N (or N-C_2) if any submodule $N' \leq N$ with $N' \cong M' \leq^{\oplus} M$ implies $N' \leq^{\oplus} N$.

Theorem

Let $\{M_i\}_{i \in \mathcal{I}}$ be a class of right R-modules where $\mathcal{I} = \{1, 2, \cdots, n\}$. Assume that M_i is M_j-C_2 for all $i, j \in \mathcal{I}$. Then T.F.A.E:

(a) $\bigoplus_{i=1}^{n} M_i$ *is a Rickart module;*

(b) M_i *is M_j-Rickart for all $i, j \in \mathcal{I}$.*

Proposition

Let $M_i \trianglelefteq \bigoplus_{i \in \mathcal{I}} M_i$, $\forall\ i \in \mathcal{I}$, $\mathcal{I}$ is an arbitrary index set. Then $\bigoplus_{i \in \mathcal{I}} M_i$ is a Rickart (respectively, Baer) module $\Leftrightarrow$ M_i is a Rickart (respectively, Baer) module, $\forall\ i \in \mathcal{I}$.

As a consequence, we get: (can also be obtained by using results of [ABT] and [RR])

Theorem

Let M be a nonsingular extending module and E be a nonsingular injective module. Then M and E are relatively Rickart to each other and $M \oplus E$ is a Baer (hence, also Rickart) module.

Corollary

Let M be a nonsingular extending module. Then M and $E(M)$ are relatively Rickart to each other and $E(M) \oplus M$ is a Rickart module. In this case, $E(M) \oplus M$ is a Baer module.

We have examples that show that neither the extending condition nor the nonsingular condition is superfluous in the previous corollary.

Remark

If M is a nonsingular extending module then $E(M)^{(n)} \oplus M$ is a Baer module for any $n \in \mathbb{N}$.

***The case of Free Baer modules**: Our next result provides a characterization of rings R for which every free right R-module is Baer.

Theorem (Rizvi-Roman, J Algebra 2009)

The following statements are equivalent for a ring R.

1. *every free right R-module is a Baer module;*
2. *every projective right R-module is a Baer module;*
3. *R is a semiprimary, hereditary (Baer) ring.*

Since condition (3) is left-right symmetric, the left-handed versions of (1) and (2) also hold.

Carl Faith, in "Embedding Torsionless Modules in Projectives", Publ. Mat. **1990**, showed that for a von Neumann regular ring R, every f.g. torsionless right R-module embeds in a free right R-module (FGTF property) iff $M_n(R)$ is a Baer ring for every $n \in \mathbb{N}$.

Our next characterization of rings R for which every f.g. free right R-module is Baer, extends the result of Carl Faith by effectively dropping the requirement of *von Neumann regularity* of the ring R. (Recall that a module M is called *torsionless* if it can be embedded in a direct product of copies of the base ring and M is said to be finitely presented if there exists a short exact sequence $0 \to K \to F \to M \to 0$ with K and F finitely generated.)

Theorem

Let R be a ring. The following statements are equivalent.

1. *every f.g. free (projective) right module over R is a Baer module;*
2. *every f.g. torsionless right R-module is projective;*
3. *R is left semihereditary and right Π-coherent (i.e. every finitely generated torsionless right R-module is finitely presented);*
4. *$M_n(R)$ is Baer ring for every $n \in \mathbb{N}$.*

In particular, a ring R satisfying these equivalent conditions is right and left semihereditary.

The next result illustrates an application to the case when the base ring R is a commutative integral domain.

It is well-known that an $n \times n$ matrix ring over a commutative integral domain R is Baer if and only if every f.g. ideal of R is invertible i.e., if R is a Prüfer domain (Kaplansky).

We obtain the following for a finite rank free module over a commutative domain.

Theorem

Let R be a commutative integral domain and M a free R-module of finite rank > 1. Then M is Baer if and only if R is a Prüfer domain.

***The case of Free Rickart modules**

The class of right hereditary rings R = *every free R-module* is Rickart:

Theorem

The following conditions are equivalent for a ring R:

(a) *every free (projective) right R-module is Rickart;*

(b) *every column finite matrix ring, $CFM(R)$, is a right Rickart ring;*

(c) *the free right R-module $R^{(R)}$ is Rickart;*

(d) *R is a right hereditary ring.*

The class of right semihereditary rings R = *every f.g. free R-module* is Rickart

Theorem

The following conditions are equivalent for a ring R:

(a) *every f.g. free (projective) right R-module is Rickart;*

(b) $\mathbb{M}_n(R)$ *is a right Rickart ring for all* $n \in \mathbb{N}$*;*

(c) $\mathbb{M}_k(R)$ *is a right semihereditary ring for some* $k \in \mathbb{N}$*;*

(d) R *is a right semihereditary ring.*

As we know, for a module, Baer $\Rightarrow$ Rickart; Rickart $\nRightarrow$ Baer. A general result as a consequence of the previous result:

Proposition

Let R be a right (semi)hereditary ring which is not a Baer ring. Then every (finitely generated) free right R-module is Rickart but not Baer.

Example

Let $A = \prod_{n=1}^{\infty} \mathbb{Z}_2$. Then the ring A is commutative, von Neumann regular, and Baer. Consider $R = \{(a_n)_{n=1}^{\infty} \in A \mid a_n \text{ is eventually constant}\}$, a subring of A. Then R is a von Neumann regular ring which is not a Baer ring. Thus, every finitely generated free right R-module is a Rickart module, but is not a Baer module.

Example

$\mathbb{Z}$ is a right hereditary ring but is not a semiprimary ring. $\mathbb{Z}^{(\mathbb{R})}$ is a Rickart $\mathbb{Z}$-module but is not a Baer $\mathbb{Z}$-module.

The class of semisimple artinian rings R = *every R-module* is Rickart:

SHEY SHEY!!

∗ SOME APPLICATIONS:

An alternate proof of a result of Small can be obtained via the theory of Rickart modules.

Proposition

For any $k \in \mathbb{N}$, R is a right hereditary ring iff $\mathbb{M}_k(R)$ is a right hereditary ring.

For a commutative domain we get:

Proposition

Let R be a commutative domain. Then the following conditions are equivalent:

(a) *every finitely generated free (projective) right R-module is Rickart;*

(b) *the free right R-module $R^{(2)}$ is a Rickart module;*

(c) *$\mathbb{M}_2(R)$ is a right Rickart ring;*

(d) *R is a Prüfer domain.*