Some questions and results related to Koethe nil ideal problem

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Koethe problem

All rings are associative not necessarily with 1.

The usual extension of a ring R by adjoining 1 to R is denoted by R^*.

To denote that L is a left ideal (an ideal) of R we write $L <_l R$ ($I \lhd R$).

Koethe problem (1930). Is every left nil ideal of an arbitrary associative ring R contained in the nil radical $Nil(R)$ of R?

Equivalent formulation: Does for every left nil ideal L of an arbitrary ring R, $LR^* \subseteq Nil(R)$?

Koethe problem has a positive solution if and only if every right nil ideal of R is contained in $Nil(R)$.

Amitsur results and problems

In 1956 Amitsur proved the following results:

1. For arbitrary ring R the Jacobson radical $J(R[x])$ is equal $I[x]$ for a nil ideal I of R.

2. If R is a finitely generated algebra over an uncountable field, then $J(R) = Nil(R)$.

He raised questions:

1. Is $J(R[x]) = Nil(R)[x]$?

2. Is $J(R[x]) = Nil(R[x])$?

3. Is $Nil(R[x]) = Nil(R)[x]$?

4. Is the Jacobson radical of every finitely generated algebra over a field nil?

It turns out that 1 is equivalent to Koethe problem. In 1972 Krempa proved the following

Theorem. The following conditions are equivalent
a) Koethe problem has a positive solution;
b) for every nil ring R, $R[x]$ is Jacobson radical;
c) for every nil ring R the ring $M_2(R)$ of 2×2-matrices over R is nil.

He also proved that Koethe problem has a positive solution if and only if it has a positive solution in the class of F-algebras over any given field F.

In 1981 Beidar constructed for every countable field F an example of a finitely generated F-algebra A such that $J(A)$ is not nil.

In 2000 Somktunowicz constructed a counterexample to 3 and in 2001 Smoktunowicz and E.P. modified it getting a counterexample to 2.

In 2002 Smoktunowicz developed her construction further and she constructed for every countable field F, a simple nil F-algebra. The problem whether there exist simple nil algebras over uncountable fields is still open.

Andrunakievich problem

Define for a given ring R, $A(R) = \sum\{L <_l R \mid L \text{ nil}\}$. It is not hard to see that $A(R) \lhd R$ and $A(R) = Nil(R)$ for every ring R if and only if Koethe problem has a positive solution.

In 1969 Andrunakievich asked whether for every ring R, $A(R/A(R)) = 0$?

It turns out that this problem is equivalent to Koethe problem (E.P., 2006).

Suppose that Koethe problem has a negative solution. Define for a given ring R the chain:

- $A_0(R) = 0$.
- Suppose that $\alpha > 0$ and $A_\beta(R)$ are defined for all $\beta < \alpha$.

(a) If α is a limit ordinal, then $A_\alpha(R) = \bigcup_{\beta<\alpha} A_\beta(R)$.

(b) If α is not a limit ordinal, then $A_\alpha(R)$ is the ideal of R containing $A_{\alpha-1}(R)$ such that $A_\alpha(R)/A_{\alpha-1}(R) = A(R/A_{\alpha-1}(R))$, or equivalently $A_\alpha(R)$ is the sum of all the left ideals L of R such that L is nil modulo $A_{\alpha-1}(R)$.

In 2010 Chebotar, P.H. Lee and E.P. proved that if Koethe problem has a negative solution, then for every α there exists a ring R such that $A_\alpha(R) = A_{\alpha+1}(R) \neq A_\beta(R)$ for $\beta < \alpha$.

In 1969 Andrunakievich asked also whether every ring R such that $A(R) = 0$ (equivalently, R contains no non-zero left nil ideal) can be homomorphically mapped onto a prime ring R' such that $A(R') = 0$. This problem is still open.

Brown-McCoy radical

A ring is called Brown-McCoy radical if and only if it cannot be homomorphically mapped onto a ring with 1.

Questions(E.P., 1993) Let R be a nil ring.

1. Is $R[x]$ Brown-McCoy radical?

2. Let X be a set of cardinality ≥ 2.

a) Is the polynomial ring $R[X]$ in commuting indeteminates from X Brown-McCoy radical?

b) Is the polynomial ring $R\langle X\rangle$ it non-commuting indeterminates from X Brown-McCoy radical?

In 1998 Smoktunowicz and E.P. proved the following

Theorem For a given ring R, $R[x]$ is Brown-McCoy radical if and only if R cannot be homomorphically mapped onto a prime ring R' such that for every $0 \neq I \lhd R'$, $Z(R') \cap I \neq 0$.

This in particular gave a positive answer to 1.

In 2002 Smoktunowicz and in 2003 Ferrero and Wisbauer proved that if X is infinite then for any (not necessarily nil) ring R, $R[X]$ is Brown-McCoy radical if and only if $R\langle X\rangle$ is Brown-McCoy radical. Moreover Smoktunowicz showed that if $R[x]$ is Jacobson radical, then $R[x, y]$ is Brown-McCoy radical.

In 2006 Chebotar, Ke, P.H. Lee and E.P. proved that if R is a nil algebra over a field of a positive characteristic, then $R[x, y]$ is Brown-McCoy radical.

In the context of these questions Beidar asked:

1. Does there exists a prime ring R with trivial center such that the central closure of R is a simple ring with 1?

2. Does there exist a prime nil ring R such that the central closure of R is a simple ring with 1?

An example answering the first of these questions was constructed by Chebotar in 2008.

Behrens radical

A ring is called Behrens radical if it cannot be homomorphically mapped onto a ring with a non-zero idempotent. It is not hard to check that a ring R is Behrens radical if and only if every left ideal of R is Brown-McCoy radical.

In 2001 Beidar, Fong and E.P. proved that for every nil ring R, $R[x]$ is Behrens radical.

In this context it is natural to ask whether the results on the Brown-McCoy radical of polynomial rings extend to the Behrens radical. It turns out that it is not quite so. In 2008 P.H. Lee and E. P. showed that there exists a ring R such that $R[X]$ is Behrens radical for every set X but $R\langle Y\rangle$ is Behrens semisimple for every set Y with cardinality ≥ 2.

It is however unknown whether for nil rings R, $R[X]$ and/or $R\langle X\rangle$ is Behrens radical.

In 2006 Smoktunowicz proved that if R is a nil ring and P is a primitive ideal of $R[x]$ (so Koethe problem has a negative solution), then $P = I[x]$ for an ideal I of R.

Some related questions on tensor product of algebras over a field F

If $F \subseteq K$ is a finite field extension, then for every F-algebra A, $A \otimes_F K$ can be embedded into $M_n(A)$ for a positive integer. Hence if A is nil and Koethe problem has a positive solution then $A \otimes_F K$ is nil as well. Thus the following question is potentially easier than Koethe problem.

Question 1. Let A be a nil F-algebra and let $F \subseteq K$ be a finite field extension. Is $A \otimes_F K$ nil?

One may also expect that this question is equivalent to Koethe problem.

As we have mentioned above it is not known whether for every nil ring R the polynomial ring $R\langle X\rangle$ in a set X of cardinality ≥ 2 of non-commuting indeterminates is Brown-McCoy radical. A positive answer to this question would also give a positive answer to the following one.

Question 2. Let A and B be nil F-algebras. Is the algebra $A \otimes_F B$ Brown-McCoy radical?

It is known (E.P., 1988) that if the field F is ordered, then $A \otimes_F A$ is nil if and only if A is locally nilpotent.

Question 3. Suppose that the field F is not ordered (say, it is the field of complex numbers or a finite field) and A is an F-algebra such that $A \otimes_F A$ is nil. Must A be locally nilpotent?

The answers of the next questions are not known even if F is ordered.
For a given F-algebra A denote by A^{op} the algebra opposite to A.

Question 4. a) Is $A \otimes A^{op}$ nil if and only if A is locally nilpotent?
b) Is $A \otimes_F A$ Jacobson radical if and only if $A \otimes_F A$ is nil?
c) Is $A \otimes_F A^{op}$ Jacobson radical if and only if $A \otimes_F A^{op}$ is nil?

It is known that if F is ordered and the tensor power of 4 copies of an F-algebra A is Jacobson radical, then A is locally nilpotent.
A positive answer to Question 4 b) would give a positive answer to the next question.

Question 5. Suppose that B is a subalgebra of an F-algebra A and $A \otimes_F A$ is Jacobson radical. Must $B \otimes_F B$ be Jacobson radical?