QUASI-BAER RING HULLS AND THEIR APPLICATIONS

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CONTENTS

1. Right Essential Overrings

1.1. Let R be a ring and S be an overring of R. Then S is called a **right essential overring** of R if R_R is essential in S_R.

1.2. Let R be a ring and S be an overring of R. Then S is called a **right ring of quotients** of R if R_R is dense in S_R.

1.3. S: a right ring of quotients of a ring $R \Rightarrow S$: a right essential overring of R.

If R is right nonsingular and S is a right essential overring of R, then S is a right ring of quotients of R.

Example 1.4. There exists a right essential overring T of a ring R, which is not a right ring of quotients of R. Take

$$R = \begin{bmatrix} \mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{bmatrix}$$

and

$$T = \begin{bmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{bmatrix}.$$

Then $(T, +, \circ)$, where $+$ is the usual addition and $\circ$ is the usual multiplication, is a right essential overring of R, but $(T, +, \circ)$ is not a right ring of quotients of R.

1.5. If $(T, +, \circ)$ and $(T, +, \bullet)$ are right ring of quotients of R, then $\circ = \bullet$.

Example 1.6. But **1.5** does not hold true for the case of right essential overrings.

In fact, let

$$R = \begin{bmatrix} \mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{bmatrix}$$

and

$$T = \begin{bmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{bmatrix}.$$

Then $R_R \leq^{\text{ess}} T_R$. The addition on T is the usual addition.

Key Idea: (i) If T is a right essential overring of R, then $1_T = 1_R$.

(ii) Thus $1_T = 1_R = e_1 + e_2$, where

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Also, $e_1 = e_1^2$, $e_2 = e_2^2$, and $e_1e_2 = e_2e_1 = 0$.

(iii) $T = e_1Te_1 + e_1Te_2 + e_2Te_1 + e_2Te_2$ and compute

$$e_1Te_1,\ e_1Te_2,\ e_2Te_1 \text{ and } e_2Te_2.$$

Assume that T has a compatible ring structure. Put $A = \mathbb{Z}_4$. By direct computation,

$$(1)\, e_1Te_1 = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix},\ (2)\, e_2Te_1 = 0, \text{ and } (3)\, e_2Te_2 = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}.$$

$$(4)\ e_1Te_2 = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \text{ or } e_1Te_2 = \left\{0, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 0 & 2 \end{bmatrix}\right\}.$$

By (1), (2), (3), and (4), we get the following Cases 1 and 2.

Case 1. $e_1Te_1 = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$, $e_1Te_2 = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$, $e_2Te_1 = 0$, and

$e_2Te_2 = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}$.

In this case, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ since $e_1 \in e_1Te_1$

and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in e_1Te_2$.

Also $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in e_1Te_2\, e_1Te_2 = 0$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in e_2\, e_1Te_2 = 0$.

So there exists a multiplication on T such that T has a compatible ring structure under this multiplication $\diamond_1$ given by

$$\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} \diamond_1 \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 & a_1b_2 + b_1c_2 \\ 0 & c_1c_2 \end{bmatrix}.$$

We remark that the ring $(T, +, \diamond_1)$ is the 2×2 upper triangular matrix ring $T_2(A)$ over the ring A.

Case 2. $e_1Te_1 = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$, $e_1Te_2 = \left\{0, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 0 & 2 \end{bmatrix}\right\}$,

$e_2Te_1 = 0$, and $e_2Te_2 = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}$.

In this case, there is another compatible ring structure on T as shown in the next steps.

Step 1. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$.

Note that $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ since $\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \in e_1Te_2$ and $e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

So

$$\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

because $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \in e_1e_2Te_2 = 0$.

Step 2. $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$.

Step 3. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$.

Step 4. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

By Steps 1, 2, 3, and 4 of Case 2, there is also a multiplication $\diamond_2$ on T such that

T has a compatible ring structure under this multiplication:

$$\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} \diamond_2 \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 & a_1b_2 + 2b_1b_2 + b_1c_2 \\ 0 & c_1c_2 + 2a_1b_2 + 2c_1b_2 \end{bmatrix}.$$

Then $(T, +, \diamond_1)$ and $(T, +, \diamond_2)$ are *all possible* compatible ring structures on T.

Define $f : (T, +, \diamond_1) \to (T, +, \diamond_2)$ by

$$f \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 2b + c \end{bmatrix}.$$

Then f is a ring isomorphism. So, $(T, +, \diamond_1) \cong (T, +, \diamond_2)$.

Theorem 1.7. (Birkenmeier, Osofsky, Park, and Rizvi) Let A be a local commutative QF-ring (self-injective artinian) with $J(A) \neq 0$. Let

$$R = \begin{bmatrix} A & A/J(A) \\ 0 & A/J(A) \end{bmatrix}.$$

Then

(i) $R = Q(R)$.

(ii) $E(R_R)$ has $|\mathrm{Soc}(A)|^2$ distinct ring structures which are right essential overrings of R.

(iii) All these rings are QF and mutually isomorphic.

1.8. We can construct a ring R for which every injective hull of R_R has infinitely many distinct compatible ring structures and these are isomorphic and QF-rings.

Let F be an infinite field and

$$R = \begin{bmatrix} \Lambda & \Lambda/J(\Lambda) \\ 0 & \Lambda/J(\Lambda) \end{bmatrix},$$

where Λ is the ring in (i) and (ii) below. Then $E(R_R)$ of R_R has $|F|$ distinct compatible ring structures. These compatible ring structures on $E(R_R)$ are isomorphic and QF.

(i) $\Lambda = F[x]/f(x)F[x]$, where $f(x) \neq 0$ is not square free by an irreducible polynomial.

(ii) $\Lambda = F[G]$ is the group algebra, where the characteristic of F is $p > 0$, p a prime integer, and G is a finite abelian group such that $p \mid |G|$.

2. An Example of Osofsky

2.1. A module M is **quasi-injective** if and only if M is fully invariant in $E(M)$.

Let M be a quasi-injective module. Then:

(C_1) Every submodule of M is essential in a direct summand of M.

(C_2) If $V \leq M$ and $V \cong N \leq^{\oplus} M$, then $V \leq^{\oplus} M$.

A module M with the condition (C_2) satisfies the following condition.

(C_3) If M_1 and M_2 are direct summands of M such that $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a direct summand of M.

Let M be a module.

(i) M is called **continuous** if it satisfies (C_1) and (C_2) conditions.

(ii) M is said to be **quasi-continuous** if it has (C_1) and (C_3) conditions.

(iii) M is called **extending** (or **CS**) if it satisfies (C_1) condition.

A module M is said to be **FI-extending** (fully invariant extending) if every fully invariant submodule is essential in a direct summand of M.

$$\text{injective} \Rightarrow \text{quasi-injective} \Rightarrow \text{continuous}$$

$$\Rightarrow \text{quasi-continuous} \Rightarrow \text{extending} \Rightarrow \text{FI-extending}$$

2.2. Say $A = \mathbb{Z}_4$ and let

$$R = \begin{bmatrix} A & 2A \\ 0 & A \end{bmatrix},$$

which is a subring of $T_2(A)$, 2×2 upper triangular matrix ring over A.

Then $R = Q(R)$. However, there are exactly 13 right essential overrings of R.

For $f \in \mathrm{Hom}(2A_A, A_A)$ and $x \in A$, let $(f \cdot x)(s) = f(xs)$ for all $s \in 2A$.

We put

$$E = \begin{bmatrix} A \oplus \mathrm{Hom}(2A_A,\, A_A) & A \\ \mathrm{Hom}(2A_A,\, A_A) & A \end{bmatrix},$$

where the addition on E is componentwise and the R-module scalar multiplication

of E over R is given by:

$$\begin{bmatrix} a+f & b \\ g & c \end{bmatrix} \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} ax + f \cdot x & ay + f(y) + bz \\ g \cdot x & g(y) + cz \end{bmatrix}$$

for $\begin{bmatrix} a+f & b \\ g & c \end{bmatrix} \in E$ and $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in R$, where $a, b, c, x, y, z \in A$ and

$f, g \in \mathrm{Hom}\,(2A_A, A_A)$.

Theorem 2.3. E is an injective hull of R_R.

Example 2.4. Therefore *all possible* intermediate R-modules between R_R and E_R are:

$$E = \begin{bmatrix} A \oplus \mathrm{Hom}(2A_A,\, A_A) & A \\ \mathrm{Hom}(2A_A,\, A_A) & A \end{bmatrix}, \quad V = \begin{bmatrix} A \oplus \mathrm{Hom}\,(2A_A, A_A) & 2A \\ \mathrm{Hom}\,(2A_A, A_A) & A \end{bmatrix},$$

$$Y = \begin{bmatrix} A \oplus \mathrm{Hom}\,(2A_A, A_A) & A \\ 0 & A \end{bmatrix}, \quad W = \begin{bmatrix} A & A \\ \mathrm{Hom}\,(2A_A, A_A) & A \end{bmatrix},$$

$$S = \begin{bmatrix} A & 2A \\ \mathrm{Hom}\,(2A_A, A_A) & A \end{bmatrix},$$

$$U = \begin{bmatrix} A \oplus \mathrm{Hom}\,(2A_A, A_A) & 2A \\ 0 & A \end{bmatrix}, \ T = \begin{bmatrix} A & A \\ 0 & A \end{bmatrix}, \text{ and } R = \begin{bmatrix} A & 2A \\ 0 & A \end{bmatrix}.$$

2.5. E, Y, and W cannot be right essential overrings of R.

Proof. Assume that E has a ring structure which is a right essential overring of R. Then

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} f_0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} f_0 & 0 \\ 0 & 0 \end{bmatrix}\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) =$$
$$\left(\begin{bmatrix} f_0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0,$$

a contradiction.

2.6. $V = \begin{bmatrix} A \oplus \mathrm{Hom}\,(2A_A, A_A) & 2A \\ \mathrm{Hom}\,(2A_A, A_A) & A \end{bmatrix}$ and $R = \begin{bmatrix} A & 2A \\ 0 & A \end{bmatrix}$, where $A = \mathbb{Z}_4$.

There are exactly **four** ring structures on V which are right essential overrings of R.

For $f \in \mathrm{Hom}(2A_A, A_A)$ and $x \in A$, let $(f \cdot x)(s) = f(xs)$ for all $s \in 2A$. Put

$$f_0 \in \mathrm{Hom}(2A_A, A_A)$$

such that

$$f_0(2a) = 2a$$

for $a \in A$. Then

$$\mathrm{Hom}(2A_A, A_A) = f_0 \cdot A.$$

Thus if $f \in \mathrm{Hom}(2A_A, A_A)$, then $f = f_0 \cdot r$ for some $r \in A$.

Key Idea: (i) If V is a right essential overring of R, then $1_V = 1_R$.

(ii) Thus $1_V = 1_R = e_1 + e_2$, where

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Also, $e_1 = e_1^2$, $e_2 = e_2^2$, and $e_1 e_2 = e_2 e_1 = 0$.

(iii) $V = e_1 V e_1 + e_1 V e_2 + e_2 V e_1 + e_2 V e_2$ and compute

$$e_1 V e_1,\ e_1 V e_2,\ e_2 V e_1,\ \text{ and } e_2 V e_2.$$

For
$$v_1 = \begin{bmatrix} a_1 + f_0 \cdot r_1 & 2b_1 \\ f_0 \cdot s_1 & c_1 \end{bmatrix},\ v_2 = \begin{bmatrix} a_2 + f_0 \cdot r_2 & 2b_2 \\ f_0 \cdot s_2 & c_2 \end{bmatrix} \text{ in } V,$$
define multiplications $\bullet_1, \bullet_2, \bullet_3$, and $\bullet_4$:

$$v_1 \bullet_1 v_2 = \begin{bmatrix} x & y \\ z & w \end{bmatrix},$$

where

$$x = a_1a_2 + f_0 \cdot r_1a_2 + f_0 \cdot a_1r_2 + f_0 \cdot r_1r_2,$$
$$y = 2a_1b_2 + 2r_1b_2 + 2b_1c_2,$$
$$z = f_0 \cdot s_1a_2 + f_0 \cdot s_1r_2 + f_0 \cdot c_1s_2,$$

and
$$w = 2s_1b_2 + c_1c_2.$$

$$v_1 = \begin{bmatrix} a_1 + f_0 \cdot r_1 & 2b_1 \\ f_0 \cdot s_1 & c_1 \end{bmatrix}, \ v_2 = \begin{bmatrix} a_2 + f_0 \cdot r_2 & 2b_2 \\ f_0 \cdot s_2 & c_2 \end{bmatrix} \text{ in } V,$$

where

$$V = \begin{bmatrix} A \oplus \operatorname{Hom}(2A_A, A_A) & 2A \\ \operatorname{Hom}(2A_A, A_A) & A \end{bmatrix}$$

$$v_1 \bullet_2 v_2 = \begin{bmatrix} x & y \\ z & w \end{bmatrix},$$

where

$$x = a_1a_2 + 2r_1r_2 + f_0 \cdot r_1a_2 + f_0 \cdot a_1r_2 + f_0 \cdot r_1r_2,$$
$$y = 2a_1b_2 + 2r_1b_2 + 2b_1c_2,$$
$$z = f_0 \cdot s_1a_2 + f_0 \cdot s_1r_2 + f_0 \cdot c_1s_2,$$

and

$$w = 2s_1b_2 + c_1c_2.$$

$$v_1 = \begin{bmatrix} a_1 + f_0 \cdot r_1 & 2b_1 \\ f_0 \cdot s_1 & c_1 \end{bmatrix}, \; v_2 = \begin{bmatrix} a_2 + f_0 \cdot r_2 & 2b_2 \\ f_0 \cdot s_2 & c_2 \end{bmatrix} \text{ in } V,$$

where

$$V = \begin{bmatrix} A \oplus \operatorname{Hom}(2A_A, A_A) & 2A \\ \operatorname{Hom}(2A_A, A_A) & A \end{bmatrix}$$

$$v_1 \bullet_3 v_2 = \begin{bmatrix} x & y \\ z & w \end{bmatrix},$$

where

$$x = a_1a_2 + 2s_1r_2 + 2a_1s_2 + 2c_1s_2 + f_0 \cdot r_1a_2 + f_0 \cdot a_1r_2 + f_0 \cdot r_1r_2,$$
$$y = 2a_1b_2 + 2r_1b_2 + 2b_1c_2,$$
$$z = f_0 \cdot s_1a_2 + f_0 \cdot s_1r_2 + f_0 \cdot c_1s_2,$$

and

$$w = 2s_1b_2 + c_1c_2.$$

$$v_1 = \begin{bmatrix} a_1 + f_0 \cdot r_1 & 2b_1 \\ f_0 \cdot s_1 & c_1 \end{bmatrix},\ v_2 = \begin{bmatrix} a_2 + f_0 \cdot r_2 & 2b_2 \\ f_0 \cdot s_2 & c_2 \end{bmatrix} \text{ in } V,$$

where

$$V = \begin{bmatrix} A \oplus \operatorname{Hom}(2A_A, A_A) & 2A \\ \operatorname{Hom}(2A_A, A_A) & A \end{bmatrix}$$

$$v_1 \bullet_4 v_2 = \begin{bmatrix} x & y \\ z & w \end{bmatrix},$$

where

$$x = a_1a_2 + 2r_1r_2 + 2s_1r_2 + 2a_1s_2 + 2c_1s_2 + f_0 \cdot r_1a_2 + f_0 \cdot a_1r_2 + f_0 \cdot r_1r_2,$$

$$y = 2a_1b_2 + 2r_1b_2 + 2b_1c_2,$$

$$z = f_0 \cdot s_1a_2 + f_0 \cdot s_1r_2 + f_0 \cdot c_1s_2,$$

and

$$w = 2s_1b_2 + c_1c_2.$$

2.7. There are exactly **four** ring structures on S which are right essential overrings of R:

$$(S,+,\circ_1),(S,+,\circ_2),\ (S,+,\circ_3),\ (S,+,\circ_4),$$

where

$$A=\mathbb{Z}_4$$

and

$$S=\begin{bmatrix} A & 2A \\ \mathrm{Hom}\,(2A_A, A_A) & A \end{bmatrix}.$$

2.8. There are exactly **two** ring structures on U which are right essential overrings of R

$$(U,+,\circledcirc_1),\ (U,+,\circledcirc_2),$$

where

$$U=\begin{bmatrix} A\oplus \mathrm{Hom}\,(2A_A, A_A) & 2A \\ 0 & A \end{bmatrix}$$

and

$$R=\begin{bmatrix} A & 2A \\ 0 & A \end{bmatrix}.$$

$$A=\mathbb{Z}_4.$$

2.9. As in **1.5** There are exactly **two** ring structures on T which are right essential overrings of R:

$$(T,+,\diamond_1),\ \ (T,+,\diamond_2),$$

where

$$T=\begin{bmatrix} A & A \\ 0 & A \end{bmatrix}$$

and

$$R=\begin{bmatrix} A & 2A \\ 0 & A \end{bmatrix}.$$

$$A=\mathbb{Z}_4.$$

$$E = \begin{bmatrix} A \oplus \mathrm{Hom}(2A_A,\, A_A) & A \\ \mathrm{Hom}(2A_A,\, A_A) & A \end{bmatrix}, \quad V = \begin{bmatrix} A \oplus \mathrm{Hom}\,(2A_A, A_A) & 2A \\ \mathrm{Hom}\,(2A_A, A_A) & A \end{bmatrix},$$

$$Y = \begin{bmatrix} A \oplus \mathrm{Hom}\,(2A_A, A_A) & A \\ 0 & A \end{bmatrix}, \quad W = \begin{bmatrix} A & A \\ \mathrm{Hom}\,(2A_A, A_A) & A \end{bmatrix},$$

$$S = \begin{bmatrix} A & 2A \\ \mathrm{Hom}\,(2A_A, A_A) & A \end{bmatrix},$$

$$U = \begin{bmatrix} A \oplus \mathrm{Hom}\,(2A_A, A_A) & 2A \\ 0 & A \end{bmatrix}, \ T = \begin{bmatrix} A & A \\ 0 & A \end{bmatrix}, \ \text{and } R = \begin{bmatrix} A & 2A \\ 0 & A \end{bmatrix}.$$

$$A = \mathbb{Z}_4.$$

2.10. (i) $(V, +, \bullet_1) \cong (V, +, \bullet_2) \cong (V, +, \bullet_3) \cong (V, +, \bullet_4)$.

(ii) $(S, +, \circ_1)$ is a subring of both $(V, +, \bullet_1)$ and $(V, +, \bullet_2)$.

(iii) $(S, +, \circ_2)$ is a subring of both $(V, +, \bullet_3)$ and $(V, +, \bullet_4)$.

(iv) $(S, +, \circ_1) \cong (S, +, \circ_2) \ncong (S, +, \circ_3) \cong (S, +, \circ_4)$.

(v) $(U, +, \circledcirc_1)$ is a subring of both $(V, +, \bullet_1)$ and $(V, +, \bullet_2)$.

(vi) $(U, +, \circledcirc_2)$ is a subring of both $(V, +, \bullet_3)$ and $(V, +, \bullet_4)$.

(vii) $(U, +, \circledcirc_2) \cong (U, +, \circledcirc_2)$.

(viii) $(T, +, \diamond_1) \cong (T, +, \diamond_2)$.

$$E = \begin{bmatrix} A \oplus \mathrm{Hom}(2A_A,\, A_A) & A \\ \mathrm{Hom}(2A_A,\, A_A) & A \end{bmatrix}, \quad V = \begin{bmatrix} A \oplus \mathrm{Hom}\,(2A_A, A_A) & 2A \\ \mathrm{Hom}\,(2A_A, A_A) & A \end{bmatrix},$$

$$Y = \begin{bmatrix} A \oplus \mathrm{Hom}\,(2A_A, A_A) & A \\ 0 & A \end{bmatrix}, \quad W = \begin{bmatrix} A & A \\ \mathrm{Hom}\,(2A_A, A_A) & A \end{bmatrix},$$

$$S = \begin{bmatrix} A & 2A \\ \mathrm{Hom}\,(2A_A, A_A) & A \end{bmatrix},$$

$$U = \begin{bmatrix} A \oplus \mathrm{Hom}\,(2A_A, A_A) & 2A \\ 0 & A \end{bmatrix}, \; T = \begin{bmatrix} A & A \\ 0 & A \end{bmatrix}, \text{ and } R = \begin{bmatrix} A & 2A \\ 0 & A \end{bmatrix}.$$

$$A = \mathbb{Z}_4.$$

2.11. (i) R is not right FI-extending.

(ii) $(V, +, \bullet_1)$ (hence $(V, +, \bullet_2)$, $(V, +, \bullet_3)$, and $(V, +, \bullet_4)$) is right extending, but not right quasi-continuous.

(iii) $(S, +, \circ_3)$ (so $(S, +, \circ_4)$) is right self-injective, while $(S, +, \circ_1)$ (so $(S, +, \circ_2)$) is not even right FI-extending.

(iv) $(U, +, \circledcirc_1)$ (hence $(U, +, \circledcirc_2)$) is right FI-extending, but not right extending.

(v) $(T, +, \diamond_1)$ (hence $(T, +, \diamond_2)$) is right FI-extending, but not right extending.

$$E = \begin{bmatrix} A \oplus \mathrm{Hom}(2A_A,\, A_A) & A \\ \mathrm{Hom}(2A_A,\, A_A) & A \end{bmatrix},\quad V = \begin{bmatrix} A \oplus \mathrm{Hom}\,(2A_A, A_A) & 2A \\ \mathrm{Hom}\,(2A_A, A_A) & A \end{bmatrix},$$

$$Y = \begin{bmatrix} A \oplus \mathrm{Hom}\,(2A_A, A_A) & A \\ 0 & A \end{bmatrix},\quad W = \begin{bmatrix} A & A \\ \mathrm{Hom}\,(2A_A, A_A) & A \end{bmatrix},$$

$$S = \begin{bmatrix} A & 2A \\ \mathrm{Hom}\,(2A_A, A_A) & A \end{bmatrix},$$

$$U = \begin{bmatrix} A \oplus \mathrm{Hom}\,(2A_A, A_A) & 2A \\ 0 & A \end{bmatrix},\quad T = \begin{bmatrix} A & A \\ 0 & A \end{bmatrix},\ \text{and } R.$$

$$A = \mathbb{Z}_4.$$

2.12. (i) All minimal right FI-extending right essential overrings ring of R are:

$$(S, +, \circ_3),\ (S, +, \circ_4), (U, +, \circledcirc_1),\ (U, +, \circledcirc_2),\ (T, +, \diamond_1), (T, +, \diamond_2).$$

(ii) All minimal right extending right essential overrings of R are:

$$(V, +, \bullet_1),\ (V, +, \bullet_2),\ (V, +, \bullet_3),\ (V, +, \bullet_4), (S, +, \circ_3), (S, +, \circ_4).$$

(iii) All minimal right quasi-continuous right essential overrings of R are:

$$(S, +, \circ_3) \text{ and } (S, +, \circ_4).$$

(iv) All minimal right continuous right essential overrings of R are:

$$(S, +, \circ_3) \text{ and } (S, +, \circ_4).$$

(v) All minimal right self-injective right essential overrings of R are:

$$(S, +, \circ_3) \text{ and } (S, +, \circ_4).$$

3. Ring Hulls

Definition 3.1. Let $\mathfrak{K}$ denote a class of rings.

(i) The smallest right ring of quotients T of a ring R which belongs to $\mathfrak{K}$ is called the $\mathfrak{K}$ **absolute to** $Q(R)$ **ring hull** of R. We denote $T = \widehat{Q}_{\mathfrak{K}}(R)$.

(ii) The smallest right essential overring S of a ring R which belongs to $\mathfrak{K}$ is called the $\mathfrak{K}$ **absolute ring hull** of R. We denote $S = Q_{\mathfrak{K}}(R)$.

(iii) A minimal right essential overring of a ring R which belongs to $\mathfrak{K}$ is called a $\mathfrak{K}$ **right ring hull** of R.

A ring R is called **right FI-extending** if R_R is FI-extending. Equivalently, every two-sided ideal of R is essential in a direct summand of R_R as a right R-module.

$$\text{right injective} \Rightarrow \text{right quasi-injective} \Rightarrow \text{right continuous}$$

$$\Rightarrow \text{right quasi-continuous} \Rightarrow \text{right extending} \Rightarrow \text{right FI-extending}$$

Example 3.2. Let F be a field and put

$$R = \begin{bmatrix} F & F \oplus F \\ 0 & F \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 & x \\ 0 & a & y \\ 0 & 0 & c \end{bmatrix} \mid a, c, x, y \in F \right\}.$$

Then R is right nonsingular, $Q(R) = \mathrm{Mat}_3(F)$, and R is not right FI-extending.

(i) Let $H_1 = \begin{bmatrix} F \oplus F & F \oplus F \\ 0 & F \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 & x \\ 0 & b & y \\ 0 & 0 & c \end{bmatrix} \mid a, b, c, x, y \in F \right\}$, and let

$$H_2 = \left\{ \begin{bmatrix} a+b & a & x \\ 0 & b & y \\ 0 & 0 & c \end{bmatrix} \mid a, b, c, x, y \in F \right\}.$$

Note that R, H_1, and H_2 are subrings of $\mathrm{Mat}_3(F)$. Define $\phi : H_1 \to H_2$ by

$$\phi \begin{bmatrix} a & 0 & x \\ 0 & b & y \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} a & a-b & x-y \\ 0 & b & y \\ 0 & 0 & c \end{bmatrix}.$$

Then ϕ is a ring isomorphism. R is not right FI-extending. H_1 is right FI-extending. Thus H_2 is right FI-extending because $H_1 \cong H_2$.

Note that there is no proper intermediate subring between R and H_1, also between R and H_2. Thus H_1 and H_2 are right FI-extending ring hulls of R.

(ii) Assume that $F = \mathbb{Z}_2$. Consider

$$H_3 = \left\{ \begin{bmatrix} a+b & b & x \\ b & a & y \\ 0 & 0 & c \end{bmatrix} \mid a, b, c, x, y \in F \right\}.$$

Then the ring H_3 is right FI-extending. Also H_3 is a right FI-extending ring hull of

R because there is no proper intermediate ring between R and H_3.

But $H_3 \ncong H_1$.

Let **FI** be the class of right FI-extending rings. Say R is a ring and $\mathcal{B}(Q(R))$ is the set of central idempotents of the ring $Q(R)$. Let $R\mathcal{B}(Q(R))$ be the subring of $Q(R)$ generated by R and $\mathcal{B}(Q(R))$, which is called the **idempotent closure** of R by Beidar and Wisbauer.

Theorem 3.3. Let R be a semiprime ring. Then:

(i) $\widehat{Q}_{\mathbf{FI}}(R)$ the right FI-extending (absolute to $Q(R)$) exists.

(ii) $\widehat{Q}_{\mathbf{FI}}(R) = R\mathcal{B}(Q(R))$.

Note that $\widehat{Q}_{\mathbf{FI}}(R)$ is the smallest right FI-extending right ring of quotients of R.

4. Quasi-Baer Ring Hulls

Theorem 4.1. (W. E. Clark) Let R be a finite dimensional algebra over an algebraically closed field F. Then the following statements are equivalent.

(i) R is a twisted semigroup F-algebra of some matrix units semigroup S containing all idempotent matrix units e_{11}, $e_{22}, \ldots, e_{nn}$.

(ii) The left annihilator of every ideal of R is generated by an idempotent and R has a finite two-sided ideal lattice.

Theorem 4.2. (W. E. Clark) Let L be a finite distributive lattice. Then there exists an artinian ring R such that:

(1) The left annihilator of any ideal of R is generated by an idempotent;

(2) The lattice L is isomorphic to the sublattice $\{\ell_R(I) \mid {}_RI \leq {}_RR\}$ of the lattice of all ideals of R.

4.3. A ring R is called **quasi-Baer** if the left annihilator of every ideal of R is generated by an idempotent of R as a left ideal.

4.4. Then following conditions are equivalent.

(i) R is a quasi-Baer ring.

(ii) For each $I \trianglelefteq R$, there exists $e^2 = e \in R$ such that $r_R(I) = eR$.

Example 4.5. (i) Every prime ring is a quasi-Baer ring.

(ii) Every Baer ring is a quasi-Baer ring.

(iv) Any semiprime right Noetherian group algebra over a field is quasi-Baer.

(v) The quasi-Baer ring property is Morita invariant.

(vi) Every piecewise domain is a quasi-Baer ring.

Theorem 4.6. Let R be a quasi-Baer ring. Then the following are quasi-Baer rings.

(i) eRe where $e^2 = e \in R$.

(ii) $R[x]$, $R[x, x^{-1}]$, and $R[[x]]$.

(iii) $T_n(R)$ is quasi-Baer for any positive integer n.

(iv) $\mathrm{CFM}_\Gamma(R), \mathrm{RFM}_\Gamma(R)$, and $\mathrm{CRFM}_\Gamma(R)$ are quasi-Baer.

(v) The endomorphism ring of any projective module over a quasi-Baer ring is a quasi-Baer ring.

4.7. A ring R is called **biregular** if for every $x \in R$ there is a central idempotent $e \in R$ such that $RxR = eR$.

4.8. (i) Boolean rings are biregular.

(ii) Simple rings are biregular.

(iii) Reduced regular rings are biregular.

(iv) ([Armendariz and Steinberg]) Right self-injective regular PI-rings are biregular.

(v) If R is a semiprime PI-ring, then $Q(R)$ is biregular.

Theorem 4.9. Let R be a biregular ring. Then the following are equivalent.

(i) R is a quasi-Baer ring.

(ii) The lattice of principal two-sided ideals of R is complete.

Theorem 4.10. (Birkenmeier, Müller, and Rizvi) Let R be a semiprime ring. Then R is quasi-Baer if and only if R is right FI-extending.

4.11. Let $\mathbf{qB}$ be the class of quasi-Baer rings. Recall that $\widehat{Q}_{\mathbf{qB}}(R)$ denotes the quasi-Baer (absolute to $Q(R)$) ring hull of R.

In other words, $\widehat{Q}_{\mathbf{qB}}(R)$ is the smallest quasi-Baer right ring pof quotients of R.

Theorem 4.12. Let R be a semiprime ring. Then:

(i) $\widehat{Q}_{\mathbf{qB}}(R)$ exists.

(ii) $\widehat{Q}_{\mathbf{qB}}(R) = \widehat{Q}_{\mathbf{FI}}(R) = R\mathcal{B}(Q(R))$.

Theorem 4.13. Let R be a semiprime ring. Then R is quasi-Baer if and only if $\mathcal{B}(Q(R)) \subseteq R$.

4.14. Let R be a semiprime ring. Then the central closure of R, the normal closure of R, $Q^s(R)$, $Q^m(R)$, and $Q(R)$ are all quasi-Baer and right FI-extending.

4.15. Let R be a semiprime ring. If R and a ring S are Morita equivalent, then $\widehat{Q}_{\mathbf{qB}}(R)$ and $\widehat{Q}_{\mathbf{qB}}(S)$ are Morita equivalent.

4.16. For a ring R, the following are equivalent.

(i) R is regular.

(ii) $R\mathcal{B}(Q(R))$ is regular.

(iii) R is semiprime and $\widehat{Q}_{\mathbf{qB}}(R)$ is regular.

4.17. Let R be a semiprime ring. Then R has the index of nilpotency at most n if and only if $Q_{\mathbf{qB}}(R)$ has the index of nilpotency at most n.

Theorem 4.17. Let R be a semiprime ring with exactly n minimal prime ideals, say $P_1, P_2, \ldots, P_n$. Then

$$\widehat{Q}_{\mathbf{qB}}(R) \cong R/P_1 \oplus R/P_1 \oplus \cdots \oplus R/P_n.$$

5. Applications to C^*-Algebras

5.1. A C^*-*algebra* A is a Banach $*$-algebra with the additional norm condition

$$||a^*a|| = ||a||^2$$

for all $a \in A$.

5.2. The algebra $B(H)$ on a Hilbert space H is a C^*-algebra. If $H = \mathbb{C}^n$, then $B(H)$ can be naturally identified with $\mathrm{Mat}_n(\mathbb{C})$, and the adjoint is the usual conjugate transpose. Any C^*-algebra is a norm closed $*$-subalgebra of the algebra of $B(H)$ on a Hilbert space H.

5.3. An involution $*$ on an algebra Q is called *positive definite* if for any finite set $\{x_i\}_{i=1}^n$ of Q, the relation $\sum_{i=1}^n x_i x_i^* = 0$ implies that $x_i = 0$. The involution on a C^*-algebra is positive definite.

Let A be a C^*-algebra. Let

$$Q^s(A) = \{q \in Q(A) \mid qI + Iq \subseteq A \text{ for some essential ideal of } A\},$$

the symmetric ring of quotients of R. Then $*$ can be uniquely extended to an involution $*$ on $Q^s(A)$, where $Q^s(A)$ is the symmetric ring of quotients of R. This involution on $Q^s(A)$ is also positive definite.

Indeed, say $\sum_{i=1}^n x_i x_i^* = 0$ for $x_i \in Q^s(A)$ and $i = 1, \dots, n$. Then there is an essential ideal I of A such that $x_i I + I x_i \subseteq A$. For each $y \in I$,

$$\sum_{i=1}^n (yx_i)(yx_i)^* = 0.$$

So $yx_i = 0$ for all i. Hence $Ix_i = 0$, and so $x_i = 0$ for all i.

5.4. Let Q be a unital complex $*$-algebra for which $*$ is positive definite.

The positive cone Q_+ of Q is the set of all elements of the form

$$\sum x_i x_i^*$$

where $\{x_i\}$ is a finite subset of Q. For $x, y \in Q_+$, define $x \leq y$ if $y - x \in Q_+$.

Then $(Q_+, \leq)$ is a partially ordered real vector space. An element $x \in Q$ is said to be *bounded* if $xx^* \leq n1$ for some positive integer n. This is equivalent to the existence of a finite subset $\{y_i\}$ of Q such that

$$xx^* + \sum yy^* = n1.$$

The set of all bounded elements of Q is denoted by Q_b. Then Q_b is a $*$-subalgebra of Q.

5.5. For a C^*-algebra A, let

$$M(A) = \{q \in Q^s(A) \mid qA + Aq \subseteq A\},$$

the algebra of all double centralizers on A which is called the **multiplier algebra** of A. Note that $M(A)$ is the largest C^*-algebra in which A is contained as a norm closed essential ideal.

5.6. The set $\mathbf{I}_{ce}$ of all norm closed essential ideals of a C^*-algebra A forms a filter directed downwards by inclusion. Indeed, let $I, J \in \mathbf{I}_{ce}$. Then $I \cap J \in \mathbf{I}_{ce}$. Also, if $I \supseteq J$, then $M(I) \subseteq M(J)$. Thus $\{M(I)\}_{I \in \mathbf{I}_{ce}}$ is ordered by inverse inclusion.

5.7. For every C^*-algebra A, there is a unique $*$-isomorphism:

$$\lim_{I \in \mathbf{I}_{ce}} M(I) \cong Q_b(A).$$

5.8. The norm closure $M_{\mathrm{loc}}(A)$ of $Q_b(A)$ is called the **local multiplier algebra** of A.

The local multiplier algebra $M_{\mathrm{loc}}(A)$ was first used by Elliott and Pedersen to show the innerness of certain $*$-automorphisms and derivations.

Theorem 5.9. $M_{\mathrm{loc}}(A)$ is a quasi-Baer ring for any C^*-algebra A.

5.10. A C^*-algebra is called an AW^*-algebra if A is Baer.

5.11. Let A be an AW^*-algebra and I a norm closed essential ideal of A. Then $M(I) = A$. Therefore $M_{\mathrm{loc}}(A) = A$.

5.12. Let A be a C^*-algebra. The C^*-subalgebra

$$\overline{A\mathrm{Cen}(Q_b(A))}$$

of $M_{\mathrm{loc}}(A)$ is called the **bounded central closure** of A.

If $A = \overline{A\mathrm{Cen}(Q_b(A))}$, then A is said to be **boundedly centrally closed**.

Theorem 5.13. Let A be a unital C^*-algebra. Then the following are equivalent.

(i) A is boundedly centrally closed.

(ii) A is quasi-Baer.

Corollary 5.14. (i) Any AW^*-algebra is boundedly centrally closed.

(ii) $M_{\mathrm{loc}}(A)$ is boundedly centrally closed for any C^*-algebra A.

Definition 5.15. Let A be a C^*-algebra. We call the smallest boundedly centrally closed C^*-subalgebra of $M_{\mathrm{loc}}(A)$ containing A the **boundedly centrally closed hull** of A.

Note that $\mathcal{B}(Q(A)) \subseteq Q_b(A)$ and so $Q_{\mathbf{qB}}(A) \subseteq Q_b(A) \subseteq M_{\mathrm{loc}}(A)$.

Theorem 5.16. We have the following for a unital C^*-algebra A.

(i) $\overline{Q_{\mathbf{qB}}(A)} = \overline{A\mathrm{Cen}(Q_b(A))}$.

(ii) $\overline{Q_{\mathbf{qB}}(A)}$ is the boundedly centrally closed hull of A.

(iii) Let B be an intermediate C^*-algebra between A and $M_{\mathrm{loc}}(A)$.

Then B is boundedly centrally closed if and only if $\mathcal{B}(Q(A)) \subseteq B$.

Theorem 5.17. Let A be a C^*-algebra and B an intermediate C^*-algebra between A and $M_{\text{loc}}(A)$. Then we have the following.

(i) $\overline{B\mathcal{B}(Q(A))} = \overline{B\text{Cen}(Q_b(B))}$.

(ii) B is boundedly centrally closed if and only if $B = \overline{B\mathcal{B}(Q(A))}$.

(iii) $\overline{A\mathcal{B}(Q(A))}$ is the boundedly centrally closed hull of A.

Corollary 5.18. Let A be a C^*-algebra and B an intermediate C^*-algebra between A and $M_{\text{loc}}(A)$. Then we have the following.

(i) $\text{Cen}(M_{\text{loc}}(B)) = \text{Cen}(M_{\text{loc}}(A))$.

(ii) $\overline{M(B)\text{Cen}(Q_b(M(B)))} = \overline{M(B)\mathcal{B}(Q(A))}$.

(iv) $M(B)$ is boundedly centrally closed if and only if $\mathcal{B}(Q(A)) \subseteq M(B)$.

5.19. Let $\{A_i\}$ be a set of C^*-algebras. By $\prod_i^{C^*} A_i$, we denote the C^*-algebra

$$\prod_i^{C^*} A_i = \left\{ (a_i) \in \prod_i A_i \mid \sup_i ||a_i|| < \infty \right\},$$

which is called the C^**-direct product* of $\{A_i\}$.

Theorem 5.20. Let A be a C^*-algebra and Λ an index set. Then the following conditions are equivalent.

(i) There exists a set of uniform ideals $\{U_i \mid i \in \Lambda\}$ of A such that $\sum_{i\in\Lambda} U_i$ is a direct sum, $\mathbb{C}U_i = U_i$ for each i, and $\ell_A(\bigoplus_{i\in\Lambda} U_i) = 0$.

(ii) The extended centroid of A is $\mathbb{C}^{|\Lambda|}$.

(iii) $M_{\mathrm{loc}}(A)$ is a C^*-direct product of $|\Lambda|$ prime C^*-algebras.

(iv) $\mathrm{Cen}(M_{\mathrm{loc}}(A))$ is a C^*-direct product of $|\Lambda|$ copies of $\mathbb{C}$.

Corollary 5.21. Let A be an AW^*-algebra and $\aleph$ a cardinality. Then the following conditions are equivalent.

(i) The extended centroid of A is $\mathbb{C}^{\aleph}$.

(ii) A is a C^*-direct product of $\aleph$ prime AW^*-algebras.

(iii) $\mathrm{Cen}(A)$ is a C^*-direct product of $\aleph$ copies of $\mathbb{C}$.

Theorem 5.22. Let A be a C^*-algebra and n a positive integer. Then the following conditions are equivalent.

(i) A has exactly n minimal prime ideals.

(ii) $Q_{\mathbf{qB}}(A^1)$ is a direct sum of n prime C^*-algebras.

(iii) The extended centroid of A is $\mathbb{C}^n$.

(iv) $M_{\mathrm{loc}}(A)$ is a direct sum of n prime C^*-algebras.

(v) $\mathrm{Cen}(M_{\mathrm{loc}}(A)) = \mathbb{C}^n$.

In this case, every boundedly centrally closed intermediate C^*-algebra between A and $M_{\mathrm{loc}}(A)$ is a direct sum of n prime C^*-algebras.

Corollary 5.23. Let A be a C^*-algebra. Then the following are equivalent.

(i) A satisfies a PI and has exactly n minimal prime ideals.

(ii) $A \cong \mathrm{Mat}_{k_1}(\mathbb{C}) \oplus \cdots \oplus \mathrm{Mat}_{k_n}(\mathbb{C})$ ($*$-isomorphic) for some positive integers $k_1, \ldots, k_n$.