Maps Acting on Some Zero Products.

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Outline

Introduction

R : Prime ring, not necessarily with an identity.

$Z(R)$: The centre of R

Q : The symmetric Martindale quotient ring of R.

C : The centre of Q, is called the extended centroid of R.

Definition

An additive map $d\colon R \to R$ is called a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in R$.

Example: $d(x) = ax - xa$, where $a \in R$, is a derivation induced by a.

Definition

An additive map $g\colon R \to R$ is called a generalized derivation if there exists a derivation d of R such that $g(xy) = g(x)y + xd(x)$ for any $x, y \in R$.

Example: $g(x) = ax$ and $g(x) = ax + xb$, where $a, b \in R$.

Definition

An additive mapping $* : R \longrightarrow R$ is called an involution if $(ab)^* = b^* a^*$ and $(a^*)^* = a$. Any involution of R can be uniquely extend to an involution of Q.

Definition

A derivation d of R is called a $*$-derivation if $d(x^*) = d(x)^*$ for any $x \in R$. A derivation d of R is called a skew $*$-derivation if $d(x^*) = -d(x)^*$ for any $x \in R$.

If $\phi(x) = \alpha x + d(x)$, where $\alpha \in Z$ and d is a derivation, then $\phi(x)y + x\phi(y) = 0$ whenever $xy = 0$.

In [CKL], Chebotar, Ke and Lee proved that the converse is true in some special cases when R has identity and possesses a nontrivial idempotent. If $\phi\colon R \to R$ is an additive map such that $\phi(x)y + x\phi(y) = 0$ whenever $xy = 0$, then $\phi(x) = \alpha x + d(x)$ for some $\alpha \in Z$ and d is a derivation of R ([CKL, Theorem 2]).

Lee generalized this result without assuming that R has identity ([Lee04, Corollary 1.2]).

Here we want to prove an analogous result on R with an involution.

If d is a $*$-derivation, then $d(x)y^* + xd(y)^* = 0$ whenever $xy^* = 0$.

Our goal is to characterize an additive map satisfies this property.

Main Result

Theorem A *Let R be a prime ring with an involution $*$. Assume R has nontrivial idempotents. If $\delta\colon R \to R$ is an additive map such that $\delta(x)y^* + x\delta(y)^* = 0$ whenever $xy^* = 0$. Then there exists a $*$-derivation $g\colon Q \to Q$ such that $\delta(xy) = \delta(x)y + xg(y)$ for any $x, y \in R$.*

Preliminaries

Let R be a prime ring with nontrivial idempotents. Let E be the additive subgroup generated by idempotents of R, and $\overline{E}$ be the subring generated by E. We begin with a useful result for maps acting on zero products.

Theorem 2.1 ([CL, Theorem 2.3]). *Let R be a prime ring with nontrivial idempotents. If $\Phi\colon R \times R \to R$ is a bi-additive map such that $\Phi(x, y) = 0$ whenever $xy = 0$. Then $\Phi(xa, y) = \Phi(x, ay)$ for any $x, y \in R$ and any $a \in \overline{E}$. In particular, there exists a nonzero ideal I of R such that $\Phi(xa, y) = \Phi(x, ay)$ for any $x, y \in R$ and any $a \in I$.*

Another lemma we will use is about a simple functional identity. In fact, it is a special case of [Br95, Lemma 4.5].

Lemma 2.2 ([Br95, Lemma 4.5]). *Let R be a prime ring. If $f, g\colon R \to R$ are additive maps such that $f(x)y = xg(y)$ for any $x, y \in R$. Then there exists $q \in Q$ such that $f(x) = xq$ and $g(x) = qx$ for any $x \in R$.*

Lemma 3.1. *There exists a nonzero ideal $I = I^*$ of R such that*

$$\delta(xa)y + xa\delta(y^*)^* = \delta(x)ay + x\delta(y^*a^*)^* \tag{3.2}$$

for any $x, y \in R$ and any $a \in I$.

Proof. Define $\Phi(x, y) = \delta(x)y + x\delta(y^*)^*$. Then for $xy = 0$ we have $x(y^*)^* = 0$, hence $\Phi(x, y) = \delta(x)(y^*)^* + x\delta(y^*)^* = 0$ by our hypothesis. In view of Theorem 2.1, there exists a nonzero ideal I of R such that $\Phi(xa, y) = \Phi(x, ay)$ for any $x, y \in R$ and any $a \in I$. This means, $\delta(xa)y + xa\delta(y^*)^* = \delta(x)ay + x\delta(y^*a^*)^*$. We may replace I by $I \cap I^*$ and just assume $I^* = I$, as asserted.

Lemma 3.2. *There exists a $*$-derivation $g\colon I \to Q$ such that $\delta(xa) = \delta(x)a + xg(a)$ for all $x \in R$ and $a \in I$.*

Proof. By Lemma 3.1 we have

$$\big(\delta(xa) - \delta(x)a\big)y = x\big(\delta(y^*a^*)^* - a\delta(y^*)^*\big) \tag{3.3}$$

for all $x, y \in R$ and $a \in I$. Applying Lemma 2.2 to (3.3), there exists an additive map $g\colon I \to Q$ such that

$$\delta(xa) - \delta(x)a = xg(a) \tag{3.4}$$

and

$$\delta(y^*a^*)^* - a\delta(y^*)^* = g(a)y. \tag{3.5}$$

Now combining (3.4) and (3.5), we get

$$\delta(xa) = \delta(x)a + xg(a) = \delta(x)a + xg(a^*)^*. \tag{3.6}$$

So $g(a^*) = g(a)^*$ for all $a \in I$. Moreover, using (3.6) to expand $\delta(xab)$ in two ways,

$$\delta(x(ab)) = \delta(x)ab + xg(ab)$$

and

$$\delta((xa)b) = \delta(xa)b + xag(b) = \delta(x)ab + xg(a)b + xag(b)$$

for all $x \in R$ and $a, b \in I$. Hence $g(ab) = g(a)b + ag(b)$ for all $a, b \in I$, as asserted.

Lemma 3.3. *g can be uniquely extended to a $*$-derivation on Q.*

Proof. Note that from (3.6) we know $Rg(I), g(I)R \subseteq R$. Hence, if we set $J = I^2$, we have $J^* = J$ and $g(J) \subseteq g(I)I + Ig(I) \subseteq R$. This means, g restricted on J is a derivation from J into R. Hence g can be uniquely extended to a derivation on Q. For any $q \in Q$, choose W to be a nonzero ideal such that $Wq \subseteq R$. Then from (3.6) we know $\delta(wq) = \delta(w)q + wg(q) = \delta(w)q + wg(q^*)^*$ for any $w \in W$. So $W(g(q) - g(q^*)^*) = 0$, and hence $g(q^*) = g(q)^*$ for any $q \in Q$.

Proof of Main Result

Proof. From Lemma 3.2 and Lemma 3.3 we know there is a $*$-derivation $g\colon Q \to Q$ and a nonzero ideal I of R with $I^* = I$, such that $\delta(xa) = \delta(x)a + xg(a)$ for any $x \in R$ and $a \in I$. Take $x, y \in R$ and $a, b \in I$, from (3.2) we can compute $\delta(xya)b + xya\delta(b^*)^*$ in two ways:

$$\delta((xy)a)b + (xy)a\delta(b^*)^* = \delta(xy)ab + xy\delta(b^*a^*)^*$$

and

$$\delta(x(ya))b + x(ya)\delta(b^*)^* = \delta(x)yab + x\delta(b^*a^*y^*)^*$$

$$= \delta(x)yab + x\big(\delta(b^*)a^*y^* + b^*g(a^*y^*)\big)^*$$

$$= \delta(x)yab + x\big(\delta(b^*)a^*y^* + b^*g(a^*)y^* + b^*a^*g(y^*)\big)^*$$

$$= \delta(x)yab + x\big(\delta(b^*a^*)y^* + b^*a^*g(y)^*\big)^*$$

$$= \delta(x)yab + xy\delta(b^*a^*)^* + xg(y)ab.$$

So $\big(\delta(xy) - \delta(x)y - xg(y)\big) I^2 = 0$, and it follows that $\delta(xy) = \delta(x)y + xg(y)$ for any $x, y \in R$. This completes the proof of our theorem.

Recall that a derivation d of R is called a skew $*$-derivation if $d(x^*) = -d(x)^*$ for any $x \in R$. Analogously to Theorem 3.4, that is, we have

Theorem B *Let R be a prime ring with an involution $*$. Assume R has nontrivial idempotents. If $\delta\colon R \to R$ is an additive map such that $\delta(x)y^* - x\delta(y)^* = 0$ whenever $xy^* = 0$. Then there exists a skew $*$-derivation $g\colon Q \to Q$ such that $\delta(xy) = \delta(x)y + xg(y)$ for any $x, y \in R$.*

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