A CONJECTURE OF BAVULA ON HOMOMORPHISMS OF THE WEYL ALGEBRA

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Abstract. In the paper *The inversion formulae for automorphisms of polynomial algebras and differential operators in prime characteristic*, J. Pure Appl. Algebra 212 (2008), no. 10, 2320–2337, see also Arxiv:math.RA/0604477, Vladimir Bavula states the following Conjecture:

(BC) Any endomorphism of a Weyl algebra (in a finite characteristic case) is a monomorphism.

The purpose of this preprint is to prove BC for A_1, show that BC is wrong for A_n when $n > 1$, and prove an analogue of BC for symplectic Poisson algebras.

The Weyl algebra A_n is an algebra over a field F generated by $2n$ elements $x_1, \dots x_n$; $y_1, \dots, y_n$ which satisfy relations $[x_i, y_j](= x_iy_j - y_jx_i) = \delta_{ij}$, $[x_i, x_j] = 0$, $[y_i, y_j] = 0$, where δ_{ij} is the Kronecker symbol and $1 \leq i, j \leq n$. Weyl algebras appeared quite some time ago and initially were considered only over fields of characteristic zero. Arguably the most famous problem related to these algebras is the Dixmier conjecture (see [D]) that any homomorphism of A_n is an automorphism if $\mathrm{char}(F) = 0$. This problem is still open even for $n = 1$.

The finite characteristic case is certainly less popular but lately appears to attract more attention because it helps to connect questions related to the Weyl algebras and to polynomial rings, e. g. to connect the famous Jacobian Conjecture with the Dixmier conjecture (see [T1], [BK], and [AE]). There is a striking difference between the zero characteristic and the finite characteristic cases. While for characteristic zero the center of A_n is just

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For the reader's convenience the proofs of all necessary lemmas are included, eliminating the need to look for them elsewhere.

the ground field and A_n is infinite-dimensional over the center, when the characteristic is not zero the center of A_n is a polynomial ring in $2n$ generators and A_n is a finite-dimensional free module over the center.

A vector space B with two bilinear operations $x \cdot y$ (a multiplication) and $\{x, y\}$ (a Poisson bracket) is called *a Poisson algebra* if B is a commutative associative algebra under $x \cdot y$, B is a Lie algebra under $\{x, y\}$, and B satisfies the Leibniz identity: $\{x, y \cdot z\} = \{x, y\} \cdot z + y \cdot \{x, z\}$.

Here we will be concerned with symplectic (Poisson) algebras PS_n. For each n the algebra PS_n is a polynomial algebra $F[x_1, \dots x_n;\ y_1, \dots, y_n]$ with the Poisson bracket defined by $\{x_i, y_j\} = \delta_{ij},\ \ \{x_i, x_j\} = 0,\ \ \{y_i, y_j\} = 0$, where δ_{ij} is the Kronecker symbol and $1 \leq i, j \leq n$. Hence $\{f, g\} = \sum_i (\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i})$.

To distinguish PS_n and A_n we will write PS_n as $F\{x_1, y_1, \dots, x_n, y_n\}$. One can think about PS_n as a commutative approximation of A_n (and of A_n as a quantization of PS_n).

It is clear that PS_n is a polynomial algebra with some additional structure. It is less clear what is A_n. Of course, we can think about a Weyl algebra as a factor algebra of a free associative algebra by the ideal I which corresponds to the relations, but it is not obvious even that $1 \notin I$ so the resulting factor algebra may be the zero algebra.

Lemma on basis. The monomials $y_1^{j_1} x_1^{i_1} \dots y_n^{j_n} x_n^{i_n}$ form a basis $\mathcal{P}$ of A_n over F.

Proof. Any monomial μ in A_n (which in this consideration may be the zero algebra) can be written as a product $\mu = \mu_1 \mu_2 \dots \mu_n$ where μ_i is a monomial of $x_i,\ y_i$ since different pairs (x_j, y_j) commute. Furthermore, since $x_i y_i = y_i x_i + 1$ any monomial in $x_i,\ y_i$ can be written as a linear combination of monomials $y_i^k x_i^l$ with coefficients in $\mathbb{Z}$ or in $\mathbb{Z}_p$ depending on the characteristic of F.

It remains to show that the monomials in $\mathcal{P}$ are linearly independent over F. This can be done by finding a homomorphic image of A_n in which the images of monomials from $\mathcal{P}$ are linearly independent.

If char$F = 0$ there is a natural representation of A_n. Consider homomorphisms X_i and Y_j of $R = F[y_1, \dots, y_n]$ defined by $X_i(r) = \frac{\partial r}{\partial y_i}$ and $Y_j(r) = y_j r$. A straightforward computation shows that $\alpha(x_i) = X_i$, $\alpha(y_j) = Y_j$ defines a homomorphism of A_n into the ring of homomorphisms of R and that the images of monomials from $\mathcal{P}$ are linearly independent.

Unfortunately this representation is not satisfactory when char$F = p \neq 0$ because then $X_i^p = 0$.

Here is a way to remedy the problem. Consider $R_n = R[z_1, \dots, z_n]$, and homomorphisms X_i, Y_j, and Z_k of R_n defined by $X_i(r) = \frac{\partial r}{\partial y_i}$, $Y_j(r) = y_j r$ and $Z_k(r) = z_k r$ for $r \in R_n$. Since Z_k commute with X_i, Y_j, and Z_l, if we replace X_i by $\widehat{X_i} = X_i + Z_i$ then $[\widehat{X_i}, Y_j] = [X_i, Y_j]$ and $[\widehat{X_i}, \widehat{X_j}] = 0$. Now, for $\sigma = \sum f_{\mathbf{ij}} Y_1^{j_1} \widehat{X}_1^{i_1} \dots Y_n^{j_n} \widehat{X}_n^{i_n}$ we have $\sigma(1) = \sum f_{\mathbf{ij}} y_1^{j_1} z_1^{i_1} \dots y_n^{j_n} z_n^{i_n}$, which is zero only if all $f_{\mathbf{ij}} = 0$. (Here $\mathbf{i}$ and $\mathbf{j}$ are multi-indices as usual.) $\square$

Let us call the presentation of an element $a \in A_n$ through the basis $\mathcal{P}$ *standard*. Further we will use only the standard presentations of elements of A_n. So A_n is isomorphic to a corresponding polynomial ring as a vector space.

Remark 1. A_n is a domain (does not have zero divisors). To see this consider a degree-lexicographic ordering of monomials in $\mathcal{P}$ first by the total degree and then by $y_1 >> x_1 >> y_2 >> x_2 \dots >> y_n >> x_n$. Then the commutation relations of A_n give $|fg| = |f||g|$ where $|h|$ for $h \in A_n \setminus 0$ denotes the largest monomial appearing in h. $\square$

If char$(F) = 0$ then BC is very easy to prove (and is well-known) both for A_n and PS_n. Suppose that φ has a non-zero kernel. Let us take a non-zero element in the kernel of φ of the smallest total degree deg possible. It is clear that $\deg(ab) = \deg(a) + \deg(b)$ for $a,\ b \in A_n$ because of the commutation relations. Consider PS_n first. If Λ is a "minimal" element of the kernel then $\{x_i, \Lambda\} = \frac{\partial \Lambda}{\partial y_i}$ and $\{y_i, \Lambda\} = -\frac{\partial \Lambda}{\partial x_i}$ should be identically zero because of the minimality of Λ. So $\frac{\partial \Lambda}{\partial y_i} = 0$ and $\frac{\partial \Lambda}{\partial x_i} = 0$ for all i. If char$(F) = 0$ this means that $\Lambda \in F$. But our homomorphism is over F, so $\Lambda = 0$. A similar proof works for A_n where the elements $[x_i, \Lambda]$ and $[y_i, \Lambda]$ should be identically zero which again shows that $\Lambda = 0$.

From now on we assume that char$(F) = p \neq 0$.

Let us start with purely computational observations.

A straightforward computation shows that $[ab, c] = [a, c]b + a[b, c]$. Therefore $[x_1^{k+1}, y_1] = [x_1^k, y_1]x_1 + x_1^k[x_1, y_1]$ and since $[x_1, y_1] = 1$ induction on k gives $[x_1^k, y_1] = kx_1^{k-1}$. Similarly, $[x_1, y_1^k] = ky_1^{k-1}$ and the index 1 can be replaced by any $i \in \{1, 2, \dots, n\}$.

Denote $[a, b]$ by $\mathrm{ad}_a(b)$. We will use that $\mathrm{ad}_a^p(B) = \mathrm{ad}_{a^p}(b)$. In order to prove it observe that $\mathrm{ad}_a(b) = (a_l - a_r)(b)$ where a_l and a_r are the operators of left and right multiplication by a. It is clear that a_l and a_r commute. So $\mathrm{ad}_a^p(b) = (a_l - a_r)^p(b) = (a_l^p - a_r^p)(b) = \mathrm{ad}_{a^p}(b)$.

Lemma on center. (a) The center $Z(A_n)$ of $A_n = F[x_1, \dots x_n;\ y_1, \dots, y_n]$ is the polynomial ring $F[x_1^p, \dots x_n^p;\ y_1^p, \dots, y_n^p]$.
(b) The Poisson center of $PS_n = F\{x_1, \dots x_n;\ y_1, \dots, y_n\}$ is the polynomial ring $F[x_1^p, \dots x_n^p;\ y_1^p, \dots, y_n^p]$.
Proof. (a) Consider x_1^p. It is clear from the definition of A_n that x_1^p commutes with all generators with a possible exception of y_1. As we observed above, $[x_1, y_1] = 1$ implies $[x_1^k, y_1] = kx_1^{k-1}$. So $[x_1^p, y_1] = px_1^{p-1} = 0$ and x_1^p is in the center of A_n. Similarly all x_j^p and y_j^p are in the center and $Z(A_n) \supseteq E$ where $E = F[x_1^p, \dots x_n^p;\ y_1^p, \dots, y_n^p]$.

Any element $a \in A_n$ can be written as $a = \sum c_{\mathbf{i},\mathbf{j}} y_1^{j_1} x_1^{i_1} \dots y_n^{j_n} x_n^{i_n} = a_0 + \sigma$ where $0 \le i_s < p$ and $0 \le j_s < p$, $c_{\mathbf{i},\mathbf{j}} \in E$, $a_0 \in E$, and σ is the sum of all monomials of a which do not belong to E. If $a \in Z(A_n)$ then $[x_1, a] = 0$. Now, $[x_1, y_1^{j_1} x_1^{i_1} \dots y_n^{j_n} x_n^{i_n}] = [x_1, y_1^{j_1}] x_1^{i_1} \dots y_n^{j_n} x_n^{i_n} = j_1 y_1^{j_1 - 1} x_1^{i_1} \dots y_n^{j_n} x_n^{i_n}$ and we have similar formulae when x_1 is replaced by any x_i or y_j. So if one of the monomials in σ is not zero we can take the commutator of a with an appropriate x_i or y_j and obtain a non-trivial linear dependence between monomials of $\mathcal{P}$ contrary to the Lemma on basis.
(b) An element a belongs to the Poisson center $Z(\mathcal{A})$ of a Poisson algebra $\mathcal{A}$ if $\{a, b\} = 0$ for all $b \in \mathcal{A}$. If $f \in Z(PS_n)$ then $\frac{\partial f}{\partial x_i} = \{f, y_i\} = \frac{\partial f}{\partial y_i} = \{x_i, f\} = 0$ for all i which is possible only if $f \in F[x_1^p, \dots x_n^p;\ y_1^p, \dots, y_n^p]$. $\square$

Nousiainen Lemma. Let φ be a homomorphism of A_n or PS_n correspondingly. Then (a) $A_n = Z(A_n)[\varphi(x_1), \dots \varphi(x_n);\ \varphi(y_1), \dots, \varphi(y_n)]$;
(b) $PS_n = Z(PS_n)[\varphi(x_1), \dots \varphi(x_n);\ \varphi(y_1), \dots, \varphi(y_n)]$.
Proof. (a) Let $E = F[x_1^p, \dots x_n^p;\ y_1^p, \dots, y_n^p]$. From the Lemma on center $Z(A_n) = E$. The algebra A_n is a finite-dimensional module over E since any element $a \in A_n$ can be written as $a = \sum c_{\mathbf{i},\mathbf{j}} y_1^{j_1} x_1^{i_1} \dots y_n^{j_n} x_n^{i_n}$ where $0 \le i_s < p$, $0 \le j_s < p$, and $c_{\mathbf{i},\mathbf{j}} \in E$. Let $K = F(x_1^p, \dots x_n^p;\ y_1^p, \dots, y_n^p)$ be the field of fractions of E and let $D_n = K[x_1, \dots x_n;\ y_1, \dots, y_n]$. Algebra D_n is a skew-field since D_n is a finite-dimensional vector space over K and D_n does not have zero divisors according to Remark 1. (Recall that $x_i^p,\ y_j^p \in K$, so any $f \in D_n$ satisfies a non-zero relation $\sum_{i=0}^N k_i f^i = 0$ where $k_i \in K$ and $N \le p^{2n}$.)

The monomials $y_1^{j_1} x_1^{i_1} \dots y_n^{j_n} x_n^{i_n}$, where $0 \le i_s < p$, $0 \le j_s < p$, are linearly independent over K. Indeed, if $\Lambda = \sum c_{\mathbf{i},\mathbf{j}} y_1^{j_1} x_1^{i_1} \dots y_n^{j_n} x_n^{i_n} = 0$

where $c_{\mathbf{i},\mathbf{j}} \in K$ then $[x_m, \Lambda] = [y_m, \Lambda] = 0$ and we can obtain from a non-trivial dependence Λ a "smaller" one. So we can invoke induction on, e. g. the sum of the total degrees of monomials in Λ.

Since any element $a \in D_n$ can be written as $a = \sum c_{\mathbf{i},\mathbf{j}} y_1^{j_1} x_1^{i_1} \dots y_n^{j_n} x_n^{i_n}$ where $0 \le i_s < p$, $0 \le j_s < p$, and $c_{\mathbf{i},\mathbf{j}} \in K$, the dimension of D_n over K is p^{2n} .

Consider now monomials $v_1^{j_1} u_1^{i_1} \dots v_n^{j_n} u_n^{i_n}$ where $u_i = \varphi(x_i)$, $v_j = \varphi(y_j)$, $0 \le i_s < p$, and $0 \le j_s < p$. Let us check that they are also linearly independent over K. If $\Lambda = \sum c_{\mathbf{i},\mathbf{j}} v_1^{j_1} u_1^{i_1} \dots v_n^{j_n} u_n^{i_n} = 0$ then $[u_m, \Lambda] = [v_m, \Lambda] = 0$ and, since the commutation relations are the same as above, we obtain from a non-trivial dependence Λ a "smaller" one.

Since there are exactly p^{2n} of these monomials, they also form a basis of D_n over K and any element $a \in A_n \subset D_n$ can be written as $a = \sum c_{\mathbf{i},\mathbf{j}} v_1^{j_1} u_1^{i_1} \dots v_n^{j_n} u_n^{i_n}$ where $c_{\mathbf{i},\mathbf{j}} \in K$.

It remains to show that all $c_{\mathbf{i},\mathbf{j}} \in E$. Order the monomials $v_1^{j_1} u_1^{i_1} \dots v_n^{j_n} u_n^{i_n}$ degree-lexicographically as in Remark 1. Let $\mu = v_1^{j_1} u_1^{i_1} \dots v_n^{j_n} u_n^{i_n}$ be the largest monomial. Then $\mathrm{ad}_{v_n}^{i_n} \mathrm{ad}_{u_n}^{j_n} \dots \mathrm{ad}_{v_1}^{i_1} \mathrm{ad}_{u_1}^{j_1}(a) = (-1)^I \prod_{m=1}^n (i_m)!(j_m)! c_{\mathbf{i},\mathbf{j}}$, where $I = \sum_{m=1}^n i_m$, belongs to A_n. Since $(-1)^I \prod_{m=1}^n (i_m)!(j_m)! \neq 0$ we conclude that $c_{\mathbf{i},\mathbf{j}} \in A_n \bigcap K = E$, replace a by $a - c_{\mathbf{i},\mathbf{j}}\mu$, and finish by induction on the number of monomials of a.

(b) Let T_n be the field of rational functions $F(x_1, \dots x_n;\ y_1, \dots, y_n)$ endowed with the same bracket as PS_n: $\{f, g\} = \sum_i (\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i})$. Then T_n becomes a Poisson algebra and we can think of PS_n as a subalgebra of T_n. It is clear that $Z(T_n) = F(x_1^p, \dots x_n^p;\ y_1^p, \dots, y_n^p)$ and that T_n is a p^{2n}-dimensional vector space over $Z(T_n)$.

Denote $u_i = \varphi(x_i)$, $v_i = \varphi(y_i)$. To show that the monomials $v_1^{j_1} u_1^{i_1} \dots v_n^{j_n} u_n^{i_n}$ where $0 \le i_s < p$ and $0 \le j_s < p$ are linearly independent over $Z(T_n)$ we, as above, can consider a "minimal" relation $\Lambda = \sum c_{\mathbf{i},\mathbf{j}} v_1^{j_1} u_1^{i_1} \dots v_n^{j_n} u_n^{i_n} = 0$ and get "smaller" relations by taking $\{u_m, \Lambda\}$ and $\{v_m, \Lambda\}$.

If $a \in PS_n$ is presented as $a = \sum c_{\mathbf{i},\mathbf{j}} v_1^{j_1} u_1^{i_1} \dots v_n^{j_n} u_n^{i_n}$ where $0 \le i_s < p$, $0 \le j_s < p$, and $c_{\mathbf{i},\mathbf{j}} \in Z(T_n)$ then all $c_{\mathbf{i},\mathbf{j}} \in Z(PS_n)$; to see this just replace, in the considerations above, ad_z by Ad_z defined by $\mathrm{Ad}_z(b) = \{z, b\}$. $\square$

Corollary. There are no homomorphisms from A_n into A_{n-1}.

Proof. Assume that we have a homomorphism $\phi : A_n \to A_{n-1}$. Consider images $u_i = \phi(x_i)$ and $v_i = \phi(y_i)$. According to Nousiainen Lemma A_{n-1} is a vector space over the center of A_{n-1} with a basis consisting of monomials $v_1^{j_1} u_1^{i_1} \dots v_{n-1}^{j_{n-1}} u_{n-1}^{i_{n-1}}$, $0 \le i_s < p$, $0 \le j_s < p$. Therefore u_n and v_n are in the center of A_{n-1} and hence commute with each other. $\square$

Remark 2. This Lemma is similar to a lemma from Pekka Nousiainen's PhD thesis (Pennsylvania State University, 1981) which was never published (see [BCW]). Nousiainen proved his Lemma in a commutative setting for a Jacobian set of polynomials, i. e. he proved that if $z_1, \ldots, z_n \in F[y_1, \ldots, y_n]$ and the Jacobian $\mathrm{J}(z_1, \ldots, z_n) = 1$ then $F[y_1, \ldots, y_n] = F[y_1^p, \ldots, y_n^p; z_1, \ldots, z_n]$. Hence $F[y_1, \ldots, y_n] = F[y_1^P, \ldots, y_n^P; z_1, \ldots, z_n]$ where $P = p^m$ and m is any natural number. Indeed, if say $y_1 = \sum c_{\mathbf{i}} z_1^{i_1} \ldots z_n^{i_n}$ where $c_{\mathbf{i}} \in F[y_1^p, \ldots, y_n^p]$ then $y_1^p = \sum c_{\mathbf{i}}^p (z_1^{i_1} \ldots z_n^{i_n})^p$ and $c_{\mathbf{i}}^p \in F[y_1^{p^2}, \ldots, y_n^{p^2}]$.

For the same reason $PS_n = Z(PS_n)^P[\varphi(x_1), \ldots \varphi(x_n);\ \varphi(y_1), \ldots, \varphi(y_n)]$. But even for A_1 the situation is different. Take e. g. $u = x,\ v = y^2x - y$ when $p = 2$. Then $A_1 \neq F[x^4, y^4; u, v]$. Indeed, $u^2 = x^2,\ v^2 = y^4x^2$ and $D_1 \neq F(x^4, y^4)[u, v]$ since u^2 and v^2 are linearly dependent over $F(x^4, y^4)$.

This difference between A_n and PS_n is the reason for BC to be correct for PS_n and wrong for A_n.

The Nousiainen Lemma for Weyl algebras is proved in [T2] and [AE]. □

Theorem 1. BC is true for Poisson algebras PS_n.

Proof. In the Nousiainen Lemma we proved that $PS_n = Z(PS_n)[u_1, \ldots u_n;\ v_1, \ldots, v_n]$ where $u_i = \varphi(x_i),\ v_i = \varphi(y_i)$. So $a = \sum c_{\mathbf{i},\mathbf{j}} v_1^{j_1} u_1^{i_1} \ldots v_n^{j_n} u_n^{i_n}$ where $c_{\mathbf{i},\mathbf{j}} \in Z(PS_n) = F[x_1^p, \ldots x_n^p;\ y_1^p, \ldots, y_n^p]$ for any $a \in PS_n$. Therefore

$$a^p = \sum c_{\mathbf{i},\mathbf{j}}^p v_1^{pj_1} u_1^{pi_1} \ldots v_n^{pj_n} u_n^{pi_n} \text{ where } c_{\mathbf{i},\mathbf{j}}^p \in F[x_1^{p^2}, \ldots, x_n^{p^2};\ y_1^{p^2}, \ldots, y_n^{p^2}].$$

Hence
$F[x_1^p, \ldots, x_n^p;\ y_1^p, \ldots, y_n^p] \subset F[x_1^{p^2}, \ldots, x_n^{p^2};\ y_1^{p^2}, \ldots, y_n^{p^2}][u_1^p, \ldots, u_n^p;\ v_1^p, \ldots, v_n^p]$
and $a = \sum d_{\mathbf{i},\mathbf{j}} v_1^{j_1} u_1^{i_1} \ldots v_n^{j_n} u_n^{i_n}$ where
$d_{\mathbf{i},\mathbf{j}} \in F[x_1^{p^2}, \ldots x_n^{p^2};\ y_1^{p^2}, \ldots, y_n^{p^2}][u_1^p, \ldots u_n^p;\ v_1^p, \ldots, v_n^p]$.

So $PS_n = F[x_1^{p^2}, \ldots, x_n^{p^2};\ y_1^{p^2}, \ldots, y_n^{p^2}][u_1, \ldots, u_n;\ v_1, \ldots, v_n]$. By iterating this process we will get that $PS_n = F[x_1^P, \ldots x_n^P;\ y_1^P, \ldots, y_n^P][u_1, \ldots, u_n;\ v_1, \ldots, v_n]$ where $P = p^m$ for any positive integer m.

It is clear that $u_i^P,\ v_j^P \in F[x_1^P, \ldots, x_n^P;\ y_1^P, \ldots, y_n^P]$ so PS_n is spanned over $F[x_1^P, \ldots, x_n^P;\ y_1^P, \ldots, y_n^P]$ by monomials $v_1^{j_1} u_1^{i_1} \ldots, v_n^{j_n} u_n^{i_n}$ where $0 \leq i_s < P$, $0 \leq j_s < P$. Of course, PS_n is spanned over $F[x_1^P, \ldots x_n^P;\ y_1^P, \ldots, y_n^P]$ by monomials $y_1^{j_1} x_1^{i_1} \ldots y_n^{j_n} x_n^{i_n}$ where $0 \leq i_s < P,\ 0 \leq j_s < P$ and these monomials are linearly independent over $F[x_1^P, \ldots x_n^P;\ y_1^P, \ldots, y_n^P]$.

So $F[x_1, \ldots x_n;\ y_1, \ldots, y_n]$ is a free module over $F[x_1^P, \ldots x_n^P;\ y_1^P, \ldots, y_n^P]$ of dimension P^{2n}.

If φ is not an injection then there is a linear dependence over F between monomials $v_1^{j_1} u_1^{i_1} \ldots v_n^{j_n} u_n^{i_n}$ where $0 \leq i_s < P,\ 0 \leq j_s < P$ provided P is sufficiently large. But this is impossible since these monomials are linearly independent over $F[x_1^P, \dots x_n^P;\ y_1^P, \dots, y_n^P]$. $\square$

We prove now using Gelfand-Kirillov dimension that a homomorphism of A_1 into A_n is an embedding. Here is a definition of the Gelfand-Kirillov dimension (GKdim) suitable for our purpose. Let R be a finitely-generated associative algebra over F: $R = F[r_1, \dots, r_m]$. Consider a free associative algebra $F_m = F\langle z_1, \dots, z_m\rangle$ and linear subspaces $F_{m,N}$ of all elements of F_m with total degree at most N. Let α be a homomorphism of F_m onto R defined by $\alpha(z_i) = r_i$ and let $R_N = \alpha(F_{m,N})$. Each R_N is a finite-dimensional vector space (over F); denote $d_N = \dim(R_N)$.

$$\mathrm{GKdim}(R) = lim_{N\to\infty}\frac{\ln(d_N)}{\ln(N)}.$$

Though this definition uses a particular system of generators, it is possible to prove that $\mathrm{GKdim}(R)$ does not depend on such a choice (see [GK] or [KL]). It is not difficult to show that in the commutative case Gelfand-Kirillov dimension coincides with the transcendence degree.

Lemma on GK-dimension. Let $R = F[z_1, \dots, z_m]$ be a ring of polynomials. If $a, b \in R$ are algebraically dependent then $\mathrm{GKdim}(S) \le 1$ for any finitely generated subalgebra $S \subset A = F[a, b]$.

Proof. If $a, b \in F$ then $F[a, b] = F$ and any subalgebra of A is F. So in this case $\mathrm{GKdim}(S) = 0$. Assume now that $a \notin F$, i. e. $\deg(a) > 0$. If $\deg(b) = 0$ then $A = F[a]$ and $d_N = N + 1$ where $d_N = \dim(A_N)$, so $\mathrm{GKdim}(A) = 1$. Let S be a subalgebra of A generated by $a_1, \dots, a_m$. Then $a_i = q_i(a)$ where q_i are polynomials. Assume that degrees of all these polynomials do not exceed d. Therefore a polynomial $g(a_1, \dots, a_m)$ of the total degree N can be rewritten as a polynomial in a of degree at most dN. Hence $N < d_N \le dN + 1$ if any of a_i is not in F and $\mathrm{GKdim}(S) = 1$. If all $a_i \in F$ then $\mathrm{GKdim}(S) = 0$.

Now, let $\deg(b) > 0$. Since a and b are algebraically dependent, $Q(a, b) = 0$ for a non-zero polynomial Q. Order monomials $a^i b^j$ by total degree $i + j$ and then lexicographically by $a >> b$. If $\mu = a^k b^l$ is the largest monomial in Q we can write $\mu = Q_1(a, b)$ where all monomials of Q_1 are less than μ. So we can replace any monomial $\nu = a^i b^j$ where $i \ge k,\ j \ge l$ by a linear combination of monomials which are less than ν. Hence any $c \in A$ of the total degree at most N can be written as a linear combination of monomials $a^i b^j$ where $i + j \le N$ and either $i < k$ or $j < l$. There are less than $(k+l)(N+1)$ and more than N monomials satisfying these properties. Therefore $N < d_N < (k+l)(N+1)$ and $\mathrm{GKdim}(A) = 1$. If S is a subalgebra of A generated by $a_1, \dots, a_m$ then $a_i = q_i(a, b)$ where q_i are polynomials of total degrees bounded by some d. If we take a polynomial $g(a_1, \dots, a_m)$

of total degree N then $g(a_1, \ldots, a_m)$ can be rewritten as a polynomial in a and b of degree at most dN. Therefore $N < d_N \leq (k+l)(dN+1)$ if any of a_i is not in F and $\mathrm{GKdim}(S) = 1$. If all $a_i \in F$ then $\mathrm{GKdim}(S) = 0$. □

Denote by deg the total degree function on A_n and by $\overline{a}$ the element of A which is deg homogeneous and such that $\deg(a - \overline{a}) < \deg(a)$. We will refer to $\overline{a}$ as the leading form of a. From the commutation relations in A_n it follows that $\overline{ab} = \overline{ba}$ and that $\deg(\overline{[a,b]}) < \deg(\overline{ab})$.

We will think about leading forms not as elements of A_n but rather as commutative polynomials. Then $\overline{ab} = \overline{a}\overline{b} = \overline{b}\overline{a}$.

Lemma on dependence. Let a and b be algebraically dependent non-zero homogeneous polynomials and $q = \deg(a)$, $r = \deg(b)$. Then a^r and b^q are proportional, i. e. there exists an $f \in F$ so that $a^r - fb^q = 0$.

Proof. The polynomials a and b are algebraically dependent, so there is a non-zero polynomial Q for which $Q(a,b) = \sum_{i,j} f_{ij} a^i b^j = 0$. Since a and b are homogeneous we may assume that $qi + rj$ is the same for all monomials of Q. Indeed, any Q can be presented as $Q = \sum_i Q_k$ where Q_k are q,r-homogeneous. Then either $Q_k(a,b) = 0$ or $\deg(Q_k(a,b)) = k$ and different components cannot cancel out.

Therefore over an algebraic closure $\overline{F}$ of F we can write $Q = f_0 a^k b^l \prod_i (a^{r'} - f_i b^{q'})$ where $f_i \in \overline{F}$, $r' = \frac{r}{d}$, $q' = \frac{q}{d}$, and d is the greatest common divisor of r and q. Hence $a^{r'} - f_i b^{q'} = 0$ for some $f_i \in \overline{F}$. But then $a^{r'} b^{-q'} \in F$ since it is a constant rational function defined over F and $a^r b^{-q} = (a^{r'} b^{-q'})^d \in F$. □

Lemma on independence. Let φ be a homomorphism of A_1 into A_n. Then the image of A_1 contains two elements with algebraically independent leading forms.

Proof. Let $u = \varphi(x)$ and $v = \varphi(y)$ where x and y are generators of A_1 and let $B = F[u,v]$ be the image of A_1 in A_n. If $\overline{u}$ and $\overline{v}$ are independent, we are done. If not, then by Lemma on dependence there exists a pair of relatively prime natural numbers (q,r) and $f \in F$ such that $\overline{u^q} = f\overline{v^r}$. Either q or r is not divisible by p. For arguments sake assume that it is q.

We can find k for which $kp + 1 \equiv 0 \pmod{q}$, $f_1 \in F$ and a positive integer s so that $\overline{u^{kp+1}} = f_1 \overline{v^s}$. Let us replace the pair (u,v) by the pair $(u_1 = u^{kp+1} - f_1 v^s, v_1 = v)$. The commutator $[u_1, v_1] = u^{kp}$ is a non-zero element of the center $Z(B)$ of B. If $\overline{u_1}$ and $\overline{v_1}$ are independent we are done, otherwise repeat the step above to get (u_2, v_2), etc.. We claim that after a finite number of steps we produce a pair of elements of B with independent leading forms.

Consider a function $\operatorname{def}(a,b) = \deg(ab) - \deg([a,b])$ on A_n. Let us check that $\operatorname{def}(u_{i+1}, v_{i+1}) < \operatorname{def}(u_i, v_i)$. We will do it for the first step since the computations are the same for every step.

Since $\overline{u^{kp+1}} = f_1\overline{v^s}$ we see that $\deg(u_1) < (kp+1)\deg(u)$. So $\operatorname{def}(u_1, v_1) = \deg(u_1v_1) - \deg([u^{kp+1} - fv^r, v]) = \deg(u_1) + \deg(v) - kp\deg(u) - \deg([u,v])) < \deg(u) + \deg(v) - \deg([u,v] = \operatorname{def}(u,v)$ since $[u^{kp+1} - fv^r, v] = u^{kp}[u,v]$ and $\deg(u_1) < (kp+1)\deg(u)$.

By definition, $\operatorname{def}(a,b) > 0$ if both a and b are not zero, so after at most $\operatorname{def}(u,v)$ steps we either get a pair with zero element or a pair $U,\ V \in B$ with independent U and V. Since $[u,v] = 1$ the pair we start with does not contain zero. Similarly, since $[u_i, v_i] \neq 0$ we cannot get a pair with zero element. $\square$

We can now see that $\operatorname{GKdim}(B) \geq 2$. Indeed, U and V are "polynomials" of u and v and we may assume that the degrees of these polynomials are at most d. Then the space of all polynomials in $u,\ v$ of degree at most N contains all polynomials in $U,\ V$ of degree at most $\frac{N}{d}$. Since $\overline{U}$ and $\overline{V}$ are algebraically independent the leading forms $\overline{U^iV^j}$ are linearly independent over F and hence U^iV^j are linearly independent over F. There are about $\frac{N^2}{2d^2}$ of these monomials with $i + j \leq \frac{N}{d}$ (exactly $\binom{[\frac{N}{d}]+2}{2}$ where $[\frac{N}{d}]$ is the integral part of $\frac{N}{d}$). So the dimension $d_N > \frac{N^2}{2d^2}$ and $\operatorname{GKdim}(B) \geq 2$.

Theorem 2. Let φ be a homomorphism of A_1 into A_n. Then φ is an embedding.

Proof. Let $A_1 = F[x;y]$ and $u = \varphi(x),\ v = \varphi(y)$. If φ has a non-zero kernel take an element a in the kernel of smallest total degree possible. Since both $[x,a]$ and $[y,a]$ are also in the kernel of φ and have smaller total degrees, $a \in Z(A_1) = F[x^p, y^p]$. Therefore u^p and v^p are algebraically dependent and by Lemma on GK-dimension $\operatorname{GKdim}(F[u^p, v^p]) \leq 1$ (recall that u^p and v^p commute). But $\operatorname{GKdim}(Z(B)) = \operatorname{GKdim}(B)$ for $B = F[u,v]$. Indeed, $B = \sum u^iv^jZ(B)$ where $0 \leq i,\ j < p$ by Lemma on center. So $d_N(B) \leq p^2 d_{\frac{N}{p}}(Z(B))$ and $d_N(B) \geq d_{\frac{N}{p}}(Z(B))$. We showed above that $\operatorname{GKdim}(B) \geq 2$. So u^p and v^p are algebraically independent and the kernel of φ consists of zero only. $\square$

Theorem 2 cannot be extended to A_2. Take $z = x_1 + y_1^{p-1}x_2,\ y_1,\ y_2$. Then $[z, y_1] = 1,\ [z, y_2] = y_1^{p-1}$, and $[z^p, y_2] = \operatorname{ad}_z^p(y_2) = \operatorname{ad}_z^{p-1}(y_1^{p-1}) = (p-1)! = -1$. For $u_1 = z + z^py_1^{p-1},\ v_1 = y_1;\ u_2 = y_2,\ v_2 = z^p$ the commutation relations of A_2 are satisfied. So ϕ which is defined by $\phi(x_i) = u_i$ and $\phi(y_i) = v_i$ is a homomorphism of A_2. If $B = \phi(A_2)$ then $B = F[u_1, u_2; v_1, v_2] =$

$F[z, y_1, y_2]$. Hence $u_1^p \in Z(B)$ and $u_1^p \in Z(F[z, y_1]) = F[z^p, y_1^p]$ since $u_1 \in F[z, y_1]$. But u_1^p should commute with $u_2 = y_2$. Therefore $u_1^p \in F[z^{p^2}, y_1^p] = F[v_2^p, v_1^p]$, u_1^p, v_1^p, v_2^p are algebraically dependent, and ϕ has a non-zero kernel.

It is an exercise to check that $\mathrm{GKdim}(B) = 3$. On the other hand, $\mathrm{GKdim}(A_2) = 4$ since $d_N = \binom{N+2n}{2n}$ for A_n which gives $\mathrm{GKdim}(A_n) = 2n$.

A question about possible GK-dimensions of images of A_n under homomorphisms seems reasonable in this setting because clearly the size of the kernel is large when the size of the image is small. Say, $\mathrm{GKdim}(\varphi(A_n)) \leq 2n$ and if φ is an injection then $\mathrm{GKdim}(\varphi(A_n)) = 2n$.

Theorem 3. $\mathrm{GKdim}(\varphi(A_n))$ can be $n + i$ where $i = 1, 2, \ldots, n$ for a homomorphism φ of A_n into A_n.

Proof. Denote $\varphi(A_n)$ by B and by $u_i = \varphi(x_i)$, $v_i = \varphi(y_i)$. It is sufficient to show that $\mathrm{GKdim}(B) = n + 1$ is possible for any n because combining $u_1, \ldots, u_m; v_1, \ldots, v_m$ of an appropriate map of A_m to A_m with $x_{m+1}, \ldots, x_n; y_{m+1}, \ldots, y_n$ we will get an image of A_n of GK-dimension $m + 1 + 2(n - m)$.

Now we shall find φ such that $\mathrm{GKdim}(B) = n + 1$.

Consider elements $z_{0,0} = 0$, $z_{m,0} = x_m - y_m^{p-1} z_{m-1,0}$ for $m = 1, \ldots, n$, and $z_{m,i} = z_{m,0}^{p^i}$. Then $[z_{i,0}, y_i] = 1$, $[z_{i-k,0}, y_i] = 0$ and $[z_{i-k,0}, x_i] = 0$ if $k > 0$, and $[z_{i,0}, z_{j,0}] = 0$. Therefore $[z_{k,i}, z_{m,j}] = 0$. We can get a relation between $z_{i,j}$ using the equality $(yx)^p - yx = y^p x^p$ if $[x, y] = 1$ (observe that $\mathrm{ad}_{yx} = \mathrm{ad}_{(yx)^p}$). Take $y_m z_{m,0} = y_m x_m - y_m^p z_{m-1,0}$. Then the summands in the right side commute and $(y_m z_{m,0})^p = (y_m x_m)^p - y_m^{p^2} z_{m-1,0}^p$. So $z_{m,0}^p = y_m^{-p}[(y_m x_m)^p - y_m^{p^2} z_{m-1,0}^p - y_m x_m + y_m^p z_{m-1,0}] = y_m^{-p}[y_m^p x_m^p - y_m^{p^2} z_{m-1,0}^p + y_m^p z_{m-1,0}] = z_{m-1,0} - y_m^{p(p-1)} z_{m-1,0}^p + x_m^p$, i. e. $z_{m,1} = z_{m-1,0} - y_m^{p(p-1)} z_{m-1,1} + x_m^p$. Since all summands in the right side of this equality commute

$$z_{m,i+1} = z_{m-1,i} - y_m^{p^{i+1}(p-1)} z_{m-1,i+1} + x_m^{p^{i+1}}$$

and

$$z_{m-1,i} = z_{m,i+1} + y_m^{p^{i+1}(p-1)} z_{m-1,i+1} - x_m^{p^{i+1}}.$$

Then by induction we can prove that

$$z_{m-1,i} = \sum_{k=0}^{m-i-2} c_{m-1,i,k} z_{m,i+1+k} + c_{m-1,i}$$

and that

$$z_{m-j,i} = \sum_{k=0}^{m-i-j-1} c_{m-j,i,k} z_{m,i+j+k} + c_{m-j,i}$$

where $c_{i,j}, c_{i,j,k} \in Z(A_n)$. The sums are finite because $z_{m,m} \in Z(A_n)$ (can be proved by induction on m starting with $z_{0,0} = 0$ since $z_{m,m} = z_{m-1,m-1} - y_m^{p^m(p-1)} z_{m-1,m} + x_m^{p^m}$).

Now we can show that all $c_{i,j,k} \in F[y_1^p, \ldots, y_2^p]$.

Since $[z_{m,i}, y_j] = [z_{m-1,i-1}, y_j] - y_m^{p^i(p-1)}[z_{m-1,i}, y_j]$, we can deduce by induction on m that $[z_{m,i}, y_j]$ is zero if $j > m-i$, one if $j = m-i$, and belongs to $F[y_1^p, \ldots, y_n^p]$ if $j < m-i$. Since $z_{m-j,i} = \sum_{k=0}^{m-i-j-1} c_{m-j,i,k} z_{m,i+j+k} + c_{m-j,i}$ we have $[z_{m-j,i}, y_l] = \sum_{k=0}^{m-i-j-1} c_{m-j,i,k}[z_{m,i+j+k}, y_l]$; this allows using $l = m-j-i, \ldots, 1$ to check that all $c_{m-j,i,k} \in F[y_1^p, \ldots, y_n^p]$.

All these computations were done to confirm that

$$z_{m,0} = \sum_{k=0}^{m-1} d_{m,k} z_{n,n-m+k} + d_m$$

where $d_m \in Z(A_n)$ and $d_{m,k} \in F[y_1^p, \ldots, y_n^p]$.

Recall that $x_m = z_{m,0} + y_m^{p-1} z_{m-1,0}$ and so

$$x_m = \sum_{k=0}^{m-1} d_{m,k} z_{n,n-m+k} + d_m + y_m^{p-1}\left(\sum_{k=0}^{m-2} d_{m-1,k} z_{n,n-m+1+k} + d_{m-1}\right).$$

Therefore

$$u_m = x_m - d_m - y_m^{p-1} d_{m-1} \in B = F[y_1, \ldots, y_n; z_{n,0}].$$

Finally, $u_1, \ldots, u_n; y_1, \ldots, y_n$ define a homomorphism of A_n into B and since any monomial in B can be written as $y_1^{j_1} \ldots y_n^{j_n} z_{n,0}^i$, the Theorem is proved. □

By looking at $\varphi(F[x_1, \ldots, x_n])$ it is possible to show that $\mathrm{GKdim}(\varphi(A_n)) \geq n$. This and Theorem 2 suggest the following

Conjecture. $\mathrm{GKdim}(\varphi(A_n)) > n$.

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