Certain Additive Maps on m-Power Closed Lie Ideals

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Notations

- R is always a prime ring, not necessarily with an identity.
- $Z(R)$: the center of R.
- Q_ℓ: the left Martindale quotient ring of R.
- C: the extended centroid of R.

Derivations

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- $d\colon R \to R$ is a derivation if
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- (Inner derivations)
  $d(x) = [a, x] = ax - xa$.
- An additive map $d\colon R \to R$ is a Jordan derivation if $d(x^2) = d(x)x + xd(x)$ for all $x \in R$.

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Theorem 1 (Herstein)

R is a prime ring of $\mathrm{char} R \neq 2$. Then any Jordan derivation is a derivation.

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Proposition 2

R is a prime ring with a nonzero left derivation.

Then R is commutative.

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Theorem 3 (Vukman and Brešar)

R is a prime ring of $\mathrm{char}(R) \neq 2, 3$. If R admits a nonzero Jordan left derivation.

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Definition 4

For a positive integer $m > 1$, a Lie ideal L of R is called **$\mathbf{m}$-power closed** *if $u^m \in L$ for all $u \in L$.*

2-power closed Lie ideal

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Theorem 5 (Awtar)

Let R be a prime ring of $\mathrm{char}(R) \neq 2$ and L be a 2-power closed Lie ideal of R.

If $d \colon R \to R$ is an additive map and is a Jordan derivation on L.

Then d is a derivation on L.

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Theorem 6 (Ashraf, Rehman and Ali)

Let R be a prime ring of $\mathrm{char}(R) \neq 2$ and L be a noncentral 2-power closed Lie ideal of R.

If d is an additive map satisfying $d(u^2) = 2ud(u)$ for all $u \in L$.

Then $d = 0$.

Main Theorem 1

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Theorem 7 (Lee and Liu)

Let R be a prime ring with $\mathrm{char}(R) = 0$ or a prime p, where $p > 2(m-1) > 1$. Suppose that L is a noncentral m-power closed Lie ideal of R.

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If $d\colon L \to R$ *is an additive map such that* $d(u^m) = mu^{m-1}d(u)$ *for all* $u \in L$, *then* $d = 0$.

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Example 1

F is a field and $m > 0$ is odd. $R \stackrel{\text{def}}{=} \mathrm{M}_2(F)$ and $L = [R, R]$.

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Example 2

Let F be a field of characteristic 2 and let $R \stackrel{\text{def}}{=} \mathrm{M}_2(F)$ and $L = [R, R]$.

Then $\dim_k R = 4$, $u^2 \in L$ for all $u \in L$ but L contains no nonzero ideals of R.

Proof

Lemma 9

Suppose that $d\colon I \to R$ is an additive map, where I is a nonzero ideal of R.
If $d(x^m) = mx^{m-1}d(x)$ for all $x \in I$, then $d = 0$.

Proof

Lemma 9

Suppose that $d\colon I \to R$ *is an additive map, where* I *is a nonzero ideal of* R.
If $d(x^m) = mx^{m-1}d(x)$ *for all* $x \in I$, *then* $d = 0$.

Sketch of Proof

Expanding $d((x+y)^m) = m(x+y)^{m-1}d(x+y)$,

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Expanding $d((x+y)^m) = m(x+y)^{m-1}d(x+y)$,and using the van der Monde argument and some replacements, $x^{2m-3}(d(x^2) - 2xd(x)) = 0$.

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By some discussion of functional identities, $d(x^2) - 2xd(x) = 0$. By Theorem 6, $d = 0$.

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Proposition 10

Suppose that $\dim_C RC = 4$ *and that* L *is a noncentral* m*-power closed Lie ideal of* R *such that* $u^{m-1} \in Z(R)$ *for all* $u \in L$*, where* m *is an odd positive integer. If* $d\colon L \to R$ *is an additive map such that* $d(u^m) = mu^{m-1}d(u)$ *for all* $u \in L$*, then* $d = 0$*.*

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For $u \in L$, $d((x+u)^m) = m(x+u)^{m-1}d(x+u)$,

so $d(u^m) = m(x+u)^{m-1}d(u)$.

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L *contains a nonzero ideal* I *such that*
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For $u \in L$, $d((x+u)^m) = m(x+u)^{m-1}d(x+u)$,
so $d(u^m) = m(x+u)^{m-1}d(u)$.
By some computations, $d(u) = 0$.

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- $d(x^2) = xd(x)$.

Theorem 11 (Zalar)

Let I be a nonzero ideal of R with $\mathrm{char}(R) \neq 2$. If $d\colon I \to R$ is an additive map such that $d(x^2) = xd(x)$ for all $x \in I$, then $d(xy) = xd(y)$ for all $x \in R$ and $y \in I$.

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R be a prime ring with $\mathrm{char}(R) = 0$ or a prime p, where $p > 2(m-1) > 1$. Suppose that L is a noncentral m-power closed Lie ideal of R.

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If $d\colon L \to R$ is an additive map such that $d(u^m) = u^{m-1}d(u)$ for all $u \in L$, then $d(u) = ua$ for some $a \in Q_\ell$, except when $\dim_C RC = 4$, m is odd, and $u^{m-1} \in Z(R)$ for all $u \in L$.

Counterexample

Example 3

$R = \mathrm{M}_2(C)$, where C is a field of $\mathrm{char}(R) = 0$ or a prime $p > 2(m-1) > 1$, where m is odd.

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$L = [R, R]$ and $d\colon L \to R$ is a C-linear map satisfying $d(e_{11}) = e_{11} + e_{12}$ and $d(e_{21}) = 0$.

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$L = [R, R]$ and $d\colon L \to R$ is a C-linear map satisfying $d(e_{11}) = e_{11} + e_{12}$ and $d(e_{21}) = 0$.
$d(u^m) = u^{m-1}d(u)$ for all $u \in L$, since $u^{m-1} \in C$ for $u \in L$.

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$L = [R, R]$ and $d\colon L \to R$ is a C-linear map satisfying $d(e_{11}) = e_{11} + e_{12}$ and $d(e_{21}) = 0$.
$d(u^m) = u^{m-1}d(u)$ for all $u \in L$, since $u^{m-1} \in C$ for $u \in L$.
However, d is not of the form $u \mapsto ua$ for some $a \in R$.

Further Results

Theorem 13 (Liu)

Let R be a prime ring with $\operatorname{char}(R) = 0$ or a prime p, where $p > 2(m+n)$, and m, n be nonnegative integers with $m + n \neq 0$.
Suppose that L is a noncentral $(m+n+1)$-power closed Lie ideal of R. If $d\colon L \to R$ is an additive map such that $d(u^{m+n+1}) = (m+n+1)u^m d(u) u^n$ for all $u \in L$, then $d = 0$.

Theorem 14 (Liu)

Let R be a prime ring with $\mathrm{char}(R) = 0$ or a prime p, where $p > 2(m+n)$, and m, n be positive integers.
Suppose that L is a noncentral $(m+n+1)$-power closed Lie ideal of R.
If $d\colon L \to R$ is an additive map such that $d(u^{m+n+1}) = u^m d(u) u^n$ for all $u \in L$,

Theorem 14 (Liu)

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Suppose that L is a noncentral $(m + n + 1)$-power closed Lie ideal of R.
If $d\colon L \to R$ is an additive map such that $d(u^{m+n+1}) = u^m d(u) u^n$ for all $u \in L$,
then $d(u) = \alpha u$ for some $\alpha \in C$,
except when $\dim_C RC = 4$, $m + n$ is even, and $u^{m+n} \in Z(R)$ for all $u \in L$.

Thank You.

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