Completely bounded disjointness preserving operators between Fourier algebras and their cb-extension

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Let G be a locally compact group, dx be the Haar measure on G.
The Banach space $L^1(G)$ is of integrable functions on G with respect to dx with the convolution product $*$, i.e.

$$f * g(y) := \int_G f(x)g(x^{-1}y)dx \quad \text{for } f, g \in L^1(G) \text{ and } y \in G.$$

$\Longrightarrow (L^1(G), *)$ is a Banach algebra which has an isometric involution defined by

$$f^*(x) := \Delta_G(x)^{-1}\overline{f(x^{-1})}, f \in L^1(G),\ x \in G,$$

where Δ_G denotes the modular function on G.

Introduction and motivation

Let G be a locally compact group, dx be the Haar measure on G. The Banach space $L^1(G)$ is of integrable functions on G with respect to dx with the convolution product $*$, i.e.

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Cohen in 1960

Let G, H be locally compact abelian groups. Let $\varphi : L^1(G) \to M(H)$ be a bounded homomorphism.
Then there is a continuous piecewise affine map $\alpha : Y \subseteq \widehat{H} \to \widehat{G}$ such that

$$\varphi u = \begin{cases} u \circ \alpha & \text{on} \quad Y, \\ 0 & \text{otherwise.} \end{cases}$$

Definition of d.p. on the group algebra

Let G, H be locally compact abelian groups. A linear map $\varphi : L^1(G) \to L^1(H)$ is called a *disjointness preserving operator* (*d.p. operator* for short) if

$$f * g = 0 \implies \varphi f * \varphi g = 0.$$

Font and Hernández (1994)

If $\varphi : L^1(G) \to L^1(H)$ is a bijective d.p. operator,

then φ is continuous and $\widehat{H}, \widehat{G}$ are homeomorphic.

We associate a map $\hat{\varphi}$ with φ by sending $\hat{f}$ to $\widehat{\varphi f}$, and denoted by $\hat{\varphi}(\hat{f})$.

If φ is a d.p. map, then we have

$$\hat{g} \cdot \hat{f} = 0 \text{ in } \widehat{L^1(G)} \implies \hat{\varphi}(\hat{g}) \cdot \hat{\varphi}(\hat{f}) = 0 \text{ in } \widehat{L^1(H)}.$$

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Positive definite functions

A function $\varphi : G \to \mathbb{C}$ is a *positive definite function* if for all $s_1, \ldots, s_n \in G$, the matrix $[\varphi(s_i^{-1}s_j)]$ is a positive definite matrix in $M_n(\mathbb{C})$.

Eymard in 1964

Let G be a locally compact group.
The *Fourier-Stieltjes algebra* $B(G)$ is the linear span of all continuous positive definite functions on G.

The *Fourier algebra* $A(G)$ is the closed ideal of $B(G)$ generated by elements with compact support.

- If G is abelian, then

$$A(G) = \{\hat{f} : f \in L^1(\hat{G})\} \quad \text{and} \quad B(G) = \{\hat{\mu} : \mu \in M(\hat{G})\}.$$

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Properties of the Fourier algebra

- $A(G)$ and $B(G)$ are semi-simple commutative Banach $*$-algebras with pointwise multiplication.
- $A(G)^* \cong vN(G)$, the von Neumann algebra generated by left regular representations of G.
- $B(G)^* \cong W^*(G)$, the enveloping von Neumann algebra generated by universal representations.
- $A(G)$ and $B(G)$ are operator spaces given by Effros and Ruan.

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Operator spaces (Effros and Ruan, 1991)

An *operator space* is linear space V with a matrix norm $\|\cdot\|$ for which

(M1) $\|v \oplus w\|_{m+n} = \max\{\|v\|_m, \|w\|_n\}$ and

(M2) $\|\alpha v \beta\|_n \leq \|\alpha\|\|v\|_m\|\beta\|$,

for all $v \in M_m(V), w \in M_n(V)$ and $\alpha \in M_{n\times m}, \beta \in M_{m\times n}$.

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Definition

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$$f \cdot g = 0 \text{ in } A(G) \quad \Longrightarrow \quad \varphi f \cdot \varphi g = 0 \text{ in } A(H).$$

- d.p. is known as a separating or zero product preserving map.
- every homomorphism is a disjointness preserving operator.

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- Abramovich (1983): every lattice homomorphism between Banach lattices is a positive d.p. operator
- for continuous functions, Arendt, Hernández, Jarosz, Kamowitz,...
- Font (1998):
  Let G, H be amenable locally compact groups. $A(G)$ and $A(H)$ are isomorphic if and only if there is a bijective d.p. operator from $A(G)$ onto $A(H)$.
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Completely bounded maps

Let $\mathcal{A}, \mathcal{B}$ be operator spaces.
A linear map $T : \mathcal{A} \to \mathcal{B}$ is called *completely bounded* if T is bounded and

$$\|T\|_{cb} := \sup\{\|T^{(n)}\| : n \in \mathbb{N}\} < \infty,$$

where $T^{(n)} : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ is given by $T^{(n)}([a_{ij}]) = [Ta_{ij}]$.

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Theorem (Ilie and Spronk, 2005)

Let G, H be locally compact groups with G amenable, and let $\varphi : A(G) \to B(H)$ be a completely bounded homomorphism. Then there is a continuous piecewise affine map $\alpha : Y \subseteq H \to G$ s.t.

$$\varphi u(h) = \begin{cases} u(\alpha(h)) & \text{if} \quad h \in Y, \\ 0 & \text{otherwise.} \end{cases}$$

Remark:
A group is called *amenable* if there exists left invariant mean on $\ell^\infty(G)$.

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Theorem

Let G, H be locally compact groups, $Y \in \Omega_0(H)$. If $\alpha : Y \to G$ is a continuous piecewise affine map and $w \in B(H)$, then the map $\varphi_{w,\alpha} : A(G) \to B(H)$ given by

$$\varphi_{w,\alpha} u(y) = \begin{cases} w(y)u(\alpha(y)) & \text{if} \quad y \in Y, \\ 0 & \text{otherwise} \end{cases}$$

is a completely bounded disjointness preserving operator.

Moreover, we can extend $\varphi_{w,\alpha}$ to a completely bounded d.p. operator $\Phi_{w,\alpha}$ on $B(G)$.
If G is amenable, then $\|\Phi_{w,\alpha}\|_{cb} = \|\varphi_{w,\alpha}\|_{cb}$.

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Theorem

Let G, H be locally compact amenable groups. If $\varphi : A(G) \to A(H)$ is a surjective c.b. d.p. operator, then

$$\varphi = w \cdot \psi_\alpha,$$

where $w \in B(H)$ is invertible and $\psi_\alpha : A(G) \to A(H)$ is a c.b. homomorphism induced by a piecewise affine proper map α.

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Corollary

If $\varphi : A(G) \to A(H)$ is a surjective c.b. d.p. operator and if H is connected, then $\|\varphi\|_{cb} = \|w\|_{B(H)}$.

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Corollary

Each surjective c.b. d.p. operator from $A(G)$ to $A(H)$ has a canonical cb-extension to $B(G)$ whenever G, H are amenable.