Preserver problems in quantum information science

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- (Dieudonné,1949) Let $\mathcal{S}$ be the set of singular matrices. An invertible linear map $L : M_n \to M_n$ satisfies $T(\mathcal{S}) \subseteq \mathcal{S}$. if and only if there are $M, N \in GL_n$ such that L has the form

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- The were also study connected to (Lie) derivations, isometry, effect algebra, Jordan homorophism, C^*-homomorphisms, etc.
- Professor Pjek-Hwee Lee and many other colleagues in the audience have nice results on this subject.

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- **So, in the study of quantum information science, one often deals with tensor products of matrices.**

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- **To determine whether $\rho \in D_{mn}$ is separable is an NP hard problem.**

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- In fact, if $C \in S_{m,n}$ then $PT_j(C) \in D_{mn}$ for $j = 1, 2$. The converse is true if $(m, n) \in \{(2, 2), (2, 3), (3, 2)\}$. Else, there are examples of entangled states with positive partial transpose.

Result for bipartite systems

Theorem [FLPS, 2010], [Flfsen & Shultz, 2010]

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where ψ_1 has the form $A \mapsto UAU^*$ or $A \mapsto UA^tU^*$,
and ψ_2 has the form $B \mapsto VBV^*$ or $B \mapsto VB^tV^*$.

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- Let $\Phi : D_{mn} \to D_{mn}$ be an affine map such that $\Phi(\mathcal{S}_{m,n}) = \mathcal{S}_{m,n}$. Then Φ can be extended uniquely to an invertible linear map $\Psi : H_{mn} \to H_{mn}$.

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- The proof of [FS] (similar to an earlier proof of ours) depends heavily on the convex feature of $\mathcal{S}_{m,n}$ and is not easy to be extended to the multi-partitite case.

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- there is $R \in P_n$ such that ψ has the form $A \mapsto (\mathrm{tr}\, A)R$.
- $m \leq n$ and there is a $U \in M_{m\times n}$ with $UU^* = I_m$ such that ψ has the form
$$A \mapsto U^*AU \quad \text{or} \quad A \mapsto U^*A^tU.$$

Extension to multi-partite systems

Theorem [FLPS,2010]

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(c) There is a permutation π on $\{1, \ldots, k\}$ and linear maps ψ_i on H_{n_i} for $i = 1, \ldots k$ such that

$$\Psi\left(\otimes_{i=1}^k A_i\right) = \otimes_{i=1}^k \psi_i\left(A_{\pi(i)}\right) \quad \text{for} \quad \otimes_{i=1}^k A_k \in \otimes_{i=1}^k P_{n_i},$$

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where ψ_i has the form

$$X \mapsto U_i X U_i^* \quad \text{or} \quad X \mapsto U_i X^t U_i^*,$$

for some unitary $U_i \in M_{n_i}$ and $n_{\pi(i)} = n_i$ for $i = 1,\ldots,k$.

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