Almost Injectivity- A survey of new and old

I am going to speak on a concept related to injective modules known as Almost Injectivity. I will give a brief survey and state some open questions.

Whereas we have a lot of literature on injective modules,and other generalizations, this concept has not caught much attention among ring theorists exceping among Japanese Ring Theorists like Harada, Oshiro, Baba.

The concept of almost injectivity is related to a number of classical rings, like, Nakayama rings, serial rings, and quasi-Frobenius rings.

PART 0

What is Almost Injective Module?

Let M and N be two right R-modules. As defined by Harada M is called almost N-injective if for each submodule X of N and each homomorphism $f : X \to M$, either there exists g such that diagram (1) commutes or there exists h such that diagram (2) commutes, where

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "i"] \arrow[d, "f"'] & N \arrow[dl, "g"] \\ & M & \end{tikzcd} \qquad (1)$$

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "i"] \arrow[d, "f"'] & N = N_1 \oplus N_2 \arrow[d, "\pi"] \\ & M \arrow[r, "h"] & N_1 \end{tikzcd} \qquad (2),$$

N_1 is a nonzero direct summand of N, and $\pi : N \to N_1$ is a projection onto N_1. Henceforth, these diagrams will be referred to as diagram (1) and diagram (2), respectively.

M is called almost self-injective if M is almost M-injective. A ring R is called right almost self-injective if it is almost self-injective as a right module over itself.

M is almost injective if it is almost injective relative to all modules N.

In particular, if N is indecomposable, then the diagram (2) gives that for every $f : X \to M$ (X, a submodule of N) there exists $h : M \to N$ such that $hf = \lambda$, where λ is inclusion map from X to N and so f must be one-to-one

ALMOST PROJECTIVITY DEFINED SIMILARLY BY REVERSING ARROWS.

Examples: (1)Every valuation domain which is not a division ring is almost self-injective but not self-injective.

(2) $Z/(p) \oplus Z/(p^2)$ as $Z-\mathrm{mod}$ule is almost self-injective but not quasi-injective. Each summand is almost injective and each summand is almost relative injective (but not relative injective) to the other,

Question 1: Whether Baer criterion holds is

not known.

An important information to record:

In order that a finite direct sum of indecomposable almost injective modules is almost injective it is necessary and sufficient that each summand is almost injective and each pair of summands is mutually almost injective.

PART I

DEFINITION. Harada defined almost right quasi-Frobenius ring as one which is right almost injective and two-sided artinian

Theorem 1: Let R be artinian.Then R is right almost QF iff every projective right module is almost injective

He gave complete characterization of almost QF rings with $J^n = 0$ for $n = 2, 3$.

This concept is preserved under Morita equivalence.

EXAMPLE: Almost Self-injectivoty versus continuity or quasi-continuity

Let D be a valuation domain which is not a division ring. Then $n \times n$ matrix ring over D ($n \geqq 2$)is almost self-injective but not continuous/quasi-continuous.

Some of the other results include

Theorem 2: Let R be an artinian ring . Then the following are equivalent:

(1) R is right almost injective.

(2) the Jacobson radical of R is almost injective as a right R-module

(3) Every projective right modle is almost injective

(4)Every f.g.projective right modle is almost injective

DEFINITION: R is called right $QF^{\#}$ ring if every injective right module is almost projective

Theorem 3: Let R be an artinian ring. Then the following are equiovalent:

(1) R is right $QF^{\#}$ ring (that is, every injective right R –module is almost projective).

(2) Every non-small right module contains a nonzero right injective module

(3) R is left almost QF(every left projective

R –modle is left almost injective)

Thus right (left) almost QF is same as left (right) almost $QF^{\#}$ under artinian.

We know if R is hereditary and QF , then R is semisimple artinian. The analoguous result for almost injectivity is

Theorem 4:Let R be an artinian ring. Then the following are equivalent:

(1) R is serial

(2) R is right almost $QF$and right almost hereditary.

(3) R is right almost $QF^{\#}$ $(= R$ is left almost $QF)$ and right almost hereditary

Remark: Did you notice unlike QF rings which are right-left symmetric, almost QF are not. Furthermore, right self-injective right noetherian is QF but if we assume noetherian instead of artinian in the defrinition of right almost QF, it need not lead to R being artinian Take for example noetherian valuation domain which is not a divsion ring.It is almost self-injective but not almost QF.

PART II

Rings without Chain Conditions:

We can show that the endomorphism ring of an indecomposable almost self-injective module is local.

Also, as stated earlier a finite direct sum of indecomposable almost self-injective modules is almost self-injective if the indecomposable modules are relatively almost injective.

NOTATION: The injective hull and the endomorphism ring of a module M will be denoted by $E(M)$ and $End(M)$, respectively. An essential submodule X of a module M will be denoted by $X \subseteq_e M$.

The following is simple but a key result

Proposition 5. An indecomposable almost self-injective module is π –injective (=quasi-continuous), hence, uniform.

COMMENT: Any homomorphism $f : X \to M$, where X is a submodule of M, with nonzero kernal (indeed for any essential kernel) can be lifted to M.

(Almost injectivity implies essential injectivity)

In addition, if M is nonsingular, then $S=End(M)$ is integral domain.

In general, $S/Z(S)$ is a domain.

Proof Let A and B be nonzero submodules of a given indecomposable almost self-injective module such that $A \cap B = 0$. Let $N = A \oplus B$. Then the projection $\pi : A \oplus B \to A$ can either be extended to an endomorphism of M (by diagram 1) or there exists an R –homomorphism g such that $g\pi = i$(by diagram 2). The latter implies $\mathrm{Ker}(\pi) = 0$, a contradiction. Hence M is π –injective.

For the last part, if $fg = 0$, then $g(M)$ is contained in $Ker f$. but $Ker f$ is closed and hence summand which yields either f or g must be zero because M is indecomposable.

For two uniform modules M and N we give below a characterization as to when M is almost N-injective in terms of their injective hulls.

Proposition *6. Let M and N be uniform modules. Then M is almost N-injective if and only if for every $f \in Hom(E(N), E(M))$ either $f(N) \subseteq M$ or f is an isomorphism and $f^{-1}(M) \subseteq N$(for the proof we need only $E(N)$ and $E(M)$to be indecom[posable)*

Proof Assume M is almost N-injective. Let $f \in Hom(E(N), E(M))$ and $X = \{n \in N|$ $f(n) \in M\}$. Then $f|_X : X \to M$. Since M is almost N-injective, either the diagram (1) or the diagram (2) holds. If (1) holds, then there exists $g : N \to M$ such that $g|_X = f|_X$. We claim $M \cap (g - f)(N) = 0$. Let $m \in M$ such that $m = (g - f)(n)$, for some $n \in N$.

Then $f(n) = g(n) - m \in M$. Hence $n \in X$. So $m = g(n) - f(n) = 0$. But $M \subseteq_e E(M)$. Hence $(g - f)(N) = 0$. That is $f(N) \subseteq M$. If (2) holds, then there exists $h : M \to N$ such that $h \circ f = 1_X$. Hence f is one to one. So f is an isomorphism since $E(N)$ is injective and $E(M)$ is an indecomposable module.
Clearly, $h|_{f(X)} = f^{-1}|_{f(X)}$. We claim $N \cap (f^{-1} - h)(M) = 0$. Let $n' \in N$ such that $n' = (f^{-1} - h)(m')$ for some $m' \in M$. Then $f^{-1}(m') = h(m') + n' \in N$. Apply f to both sides, we get $m' = ff^{-1}(m') = f(h(m') + n')$ which implies $m' \in f(X)$. So $n' = (f^{-1} - h)(m') = 0$ because $h|_{f(X)} = f^{-1}|_{f(X)}$ and $m' \in f(X)$. Hence our claim is true. Since $N \subseteq_e E(N)$,

$(f^{-1} - h)(M) = 0$. That means $f^{-1}(M) = h(M) \subseteq N$.

The converse is clear.

Suggested Problem:

If R is a semiperfect nonlocal ring which is almost self-injective, then$R$ is a direct sum of indecomposable right ideals e_iR . These e_iR are mutually almopst self-injective. Thus for each $f : E(e_iR) \to E(e_jR)$ either $f(e_iR) \subseteq e_jR$ or $E(e_iR) \cong E(e_jR)$.The latter need not imply $e_iR \cong e_jR$. This will be true if $e_iR \oplus e_jR$ were quasi-continuous.

Question 2: What will this reflect on the description of the semiperfect almost self-injective ring? Assume first radical is nil.

As an application of above theorem we have

Proposition 7. *Let R be a ring with no nontrivial idempotent.* Then *R is right almost self-injective if and only if for every $c \in E(R_R)$, either $c \in R$ or* there exists *$r \in R$ such that $cr = 1$.*

Proof Assume first R is right almost self-injective. Then R_R is uniform by above Lemma. Let $c \in E(R_R)$ and $l_c : R \longrightarrow E(R_R)$ be the left multiplication homomorphism. Then there exists $f : E(R_R) \longrightarrow E(R_R)$ such that $l_c|_R = f|_R$. By above Proposition 6 either $f(R) \subseteq R$ or f is an isomorphism and $f^{-1}(R) \subseteq R$. If

$f(R) \subseteq R$, then $c \in R$ because $f(1) = c$. If f is an isomorphism and $f^{-1}(R) \subseteq R$, then $f^{-1}(1) \in R$ and so there exists $r \in R$ such that $f(r) = 1$. So, $cr = l_c(r) = f(r) = 1$.

Conversely, suppose for every $c \in E(R_R)$, either $c \in R$ or there exists $r \in R$ such that $cr = 1$. We claim that $E(R_R)$ is uniform. For if $e \in End(E(R_R))$ is an idempotent, then either $e(1) \in R$ or there exists $r \in R$ such that $e(1)r = 1$. If $e(1) \in R$, then $e(1)$ is an idempotent in R and by assumption $e(1) = 0$ or $e(1) = 1$. Hence $e = 0$ or $e = 1_{E(R_R)}$ because $R \subseteq_e E(R_R)$. If $e(1)r = 1$ for some $r \in R$, then $e(r) = 1$. So $e(1) = e(e(r)) = e^2(r) = e(r) = 1$. So $e|_{R_R} = 1_{R_R}$. We proceed to show that $e = 1_{E(R_R)}$. Else, there exists $x \in E(R_R)$ such that $e(x) \neq x$, then $ex - x \neq 0$. Since $R \subseteq_e E(R_R)$, there exists $r' \in R$ such that $(ex - x)r' \neq 0$ and $(ex - x)r' \in R$. Because $(ex - x)r' \in R$, $(ex - x)r' = e(ex - x)r' = 0$, a contradiction to the fact that $(ex - x)r' \neq 0$. Therefore, $e = 1_{E(R_R)}$. This proves $E(R_R)$ is indecomposable and hence uniform. Thus, R_R is uniform. Now let $f \in End(E(R_R))$. Then by assumption either $f(1) \in R$ or $f(r) = 1$ for some $r \in R$. $f(1) \in R$ implies $f(R) \subseteq R$. If $f(r) = 1$ for some $r \in R$, then $f|_{rR} : rR \longrightarrow R$ is an isomorphism. Because $E(R_R)$ is uniform and injective, f is an isomorphism on $E(R_R)$ and $f^{-1}(R) = rR \subseteq R$. By above Proposition R is almost self-injective.

Corollary *8. Let D be a domain and Q its maximal right ring of quotient. Then D is right almost self-injective if and only if for every $c \in Q$, either c or $c^{-1} \in D$.This is true iff D is a valuation domain.*

Question 3: Can we give a almost injective hull of D that sits inside its injective hull Q ? or for that matter for any ring or module?

PART III

It is known that the endomorphism ring of an indecomposable quasi-injective (more generally continuous) module is local. We prove an analogous result for indecomposable almost self-injective module. Note that indecomosable almost self-injective being π –injective does not imply that its endomporphism ring must be local because this propoerty oes not hold for π –injectivity. Seemingly there is much more than π –injectoivity for indecomposable almost self-injective modules.

Theorem *9. If M is an indecomposable almost self-injective module, then $End(M)$ is local.*

For a proof of this theorem, we first prove the following interesting lemma which is of independent interest also.

Lemma *10. Let M be an indecomposable almost self-injective right module. Then for every f, $g \in S = End(M)$, (i) if $ker(f) \subsetneq ker(g)$, then $Sg \subsetneq Sf$, (ii) if $ker(f) = ker(g)$, then $Sf \subseteq Sg$ or $Sg \subseteq Sf$.*

Proof Let $ker(f) \subsetneq ker(g)$. Define $\varphi : f(M) \to g(M)$ by $\varphi(f(m)) = g(m)$. Clearly, φ is a well defined R-homomorphism. Since $ker(f) \subsetneq ker(g)$,

φ is not a one- to- one map and therefore, by assumption using diagram (1) , φ can be extended to $\psi \in S$. Hence there exists $\psi \in S$ such that $\psi(f(m)) = \varphi(f(m))$ for every $m \in M$. Thus $g(m) == \varphi(f(m) = (\psi \circ f)(m)$ for every $m \in M$. Consequently $Sg \subseteq Sf$. We prove $Sg \neq Sf$. Now $Sg = Sf$ yields $f = tg$, and $g = vf$ for some $v,t \in S$. This implies $Ker(f) = Ker(g)$, a contradiction.

Next let (ii) Let $ker(f) = ker(g)$. In this case φ is one to one. So either φ can be extended to an endomorphism $\psi \in S$ or there exists $\eta \in S$ such that $\eta \circ \varphi = 1_{f(M)}$. If $\varphi = \psi$ on $f(M)$, then as above $Sg \subseteq Sf$. If $\eta \circ \varphi = 1_{f(M)}$, then $f(m) = (\eta \circ \varphi)(f(m)) = \eta(\varphi(f(m))) = \eta(g($ for every $m \in M$. Thus $Sf \subseteq Sg$.

Corollary *11. Let M be a uniserial almost self-injective right R-module. Then End*(M) *is left uniserial.*

Let us recall the proof that if R is a right self-injective right uniserial then it is left uniserial.

Let $a, b \in R$. By Ikeda-Nakayama,

$lr(Ra) = Ra.$ By hypothesis $r(Ra)$ and $r(Rb)$ are linearly ordered and let $r(Ra) \sqsubseteq r(Rb)$. But then $lr(Ra) \sqsupseteq lr(rb)$.

This implies $Ra \sqsupseteq Rb.$ This proves our assertion for self-injective rings.The proof for almost selfinjective as above is different.

Question 4 : Is there an analogue of Ikeda-Nakayama for right almost self-injective rings?

PART IV

Since the direct sum of CS module is not necessarily CS, it has been a subject of active research to find conditions as to when the direct sum of an indecomposable family of CS modules is CS. The following remark gives one such condition in terms of almost injectivity.

Theorem 12 .*For a module M which can be* *expressed as a finite direct sum of indecomposable modules* $\{M_i\}_{i=1}^{n}$ *that the following are equivalent: (i)* $M \oplus M$ *is CS and* $End(M_i)$ *is local for each* i*; (ii)* M *is finitely* $\sum$*-CS and* $End(M_i)$ *is local for each* i*; (iii)* M_i *is almost* M_j*-injective for every* i *and* j.

Remark *In particular, If* R *is local then* R *is almost selfinjective iff* $R \times R$ *is CS. In addition,if radical is nil or* $Z(R) = J(R)$*, then* R *is selfinjecive.*

More generally, we can prove a result that answers a question whether for a right almost self-injective ring,

$$J(R) = Z(R)?$$

Theorem 13. Let R *be semiperfect. Then* R_R *is almost self-injective with* $J(R) = Z(R)$ *iff* R_R *is selfinjective.*

It is shown in the following example that a CS module with local endomorphism ring need not be almost self-injective and hence need not be finitely $\sum$-CS.

Example *14. Let* $F = \mathbb{Q}(x_1, x_2, \ldots, x_n \ldots)$*,* $S = \mathbb{Q}(x_1^2, x_2^2, \ldots, x_n^2 \ldots)$*, and*

$A = \begin{pmatrix} F & 0 \\ F & S \end{pmatrix}$. Let f be the ring homomorphism $f(a) = a$ for all $a \in \mathrm{Q}$ and $f(x_i) = x_i^2$. Let $R = \left\{ \begin{pmatrix} k & 0 \\ k' & f(k) \end{pmatrix} \mid k, k' \in F \right\}$. Then R is a subring of A. The only nontrivial right ideal of R is $\begin{pmatrix} 0 & 0 \\ F & 0 \end{pmatrix}$. Thus R is a local right uniserial (hence right CS) ring. If R is right almost self-injective then by Corollary stated earlier, R is left uniserial which is not true. Therefore, R is not right almost self-injective.

PART V

Theorem *15. Let M be a nonsingular indecomposable almost selfinjective module, $S = End(M)$, and $Q = End(E(M))$. Then*

S is a local domain and Q is its maximaal right quotient ring. Furthermore, *the following are equivalent:*

- *(i)For every $f \in Q$ either f or $f^{-1} \in S$;*
  (ii) S is a left valuation and right ore domain;
  (iii)S is right almost self-injective;
  (iv)$S \oplus S$ is CS as a right S-module;
  (v) S is finitely $\sum$-CS as a right S-module;
  (vi)S is Utumi, local and right semihereditary;
  (vii)Left side versions of (iii)-(vii).

Corollary *16. Let D be a domain. D is two-sided valuation if and only if it is left valuation and right ore if and only if it is right or left almost self-injective.*

Questions

5. If R is almost self-injective, is $R/J(R)$ also?

6. In continuation of problem 3, is $R/J(R)$regular?

Conjecture: no

7. If R is almost self-injective, any relation between $Z(R)$ and $J(R)$. Note they are equal for self-injective rings.

8. Describe structure of rings whose cyclics are $\pi - injective$.

9. What is structure of semiperfect rings rings each of whose right ideal is almost injective. Remember indecomposable almost injective is π –injective and finite direct sum of π –injectives is π –injective (continuous) if each summand is relative injective to every other summand .

10. It can be shown that every local almost selfinjective group algebra KG is selfinjective and hence G is finite. What about for semiperfect rings or, for any ring?

(Let $R = KG$ be almost selfinjective. Now KG is local. Since KG is almost selfinjective,

KG is quasi-continuous and hence CS. Thus,

$charK = p$ and G is a locally

finite p-group. Since R is indecomposable as R-module, $R \times R$ is almost self-injective R –module.and hence CS . Recall if R is local, $J(R)$ is nil and if $R \times R$ is CS then $R \times R$ satisfies the condiion $C_3$and so $R \times R$ is quasi-continuous. This proves $R = KG$ is selfinjective.

(11) If M_R is almost self-injective and

$E = End(M_R)$, is $E/J(E)$ Von Neumann regular? (This is true if M_R is injective).)

(12) If M_R is almost self-injective and $E = End(M_R)$, is $J(E) = Z(E)$?

Conjecture: No .See Theorem 13 above.

(13) If M_R is almost self-injective and $E = End(M_R)$, is $E/J(E)$ right almost self-injective? (This is true if M_R is injective.

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