On Formal Power Series Over Rickart Rings

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Dedicate to Professor P.-H. Lee

Abstract

A ring R is called a (right) Rickart ring if the right annihilator of any element in R is generated, as a right ideal, by an idempotent. This definition is equivalent to that every principle right ideal is projective, and thus a (right) Rickart ring is also known as a (right) PP ring.

Armendariz and Jondrup had shown that if R is a reduced or commutative ring, then the polynomial ring $(R[x], +, \cdot)$ is a PP ring if and only if R is a PP ring. However, this result is not true if the polynomial ring is replaced by the formal power series ring $(R[[x]], +, \cdot)$.

Birkenmeier, Kim and Park had introduced (right) principally quasi-Baer rings as a generalized for Rickart rings. A ring R is callled (right) p.q.-Baer if the right annihilator of a principal right ideal is generated by an idempotent. They also shown that: A ring R is right p.q.-Baer if and only if the polynomial ring $(R[x], +, \cdot)$ is right p.q.-Baer. Again, this result is not true if the polynomial ring is replaced by the formal power series ring $(R[[x]], +, .)$.

In this note, we discuss the conditions that guarantee the Rickart or quasi-Rickart conditions be extended to the foraml power series ring $(R[[x]], +, \cdot)$ or the nearring $(R_0[[x]], +, \circ)$.

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1st. approach

Introduction and the motivation

The study of Rickart rings has its roots in both functional analysis and homological algebra. In [37] Rickart studied C^*-algebras with the property that every right annihilator of any element is generated by a projection (an idempotent p is called a *projection* if $p = p^*$ where $*$ is an involution on that algebra). This condition is modified by Kaplansky [29] through introducing Baer rings (a ring is called a *Baer* ring if the right annihilator of every nonempty subset of R is generated, as a right ideal, by an idempotent of R) to abstract various properties of AW^*-algebras and von Neumann algebras. See also Berberian [2] for details.

A ring satisfying a generalization of Rickart's condition (i.e., every right annihilator of any element in R is generated, as a right ideal, by an idempotent) has a homological characterization as a right PP ring, i.e., every principal right ideal is projective. Left PP rings are defined similarly. A ring R is called a *Rickart* ring [2, p.18] if it is both right and left PP. Note that a Rickart ring is often refered as a PP ring in literatures. Be aware that PP rings are not initiated as a generalization for Baer rings but instead, a natural derivative from the study of torsion theory. In [21] Hattori investigated two fundamental problems in torsion theory: Is the torsion-freeness of a right module equivalent to the vanishing of its torsion part? Is it possible to divide any right module into its torsion part? The answer is affirmative, it is necessary and sufficient that the ring R is a left PP ring. In a supplement to Hattori's result, Endo [18] shows that a normal ring R (i.e., any idempotent of R is contained in the center of R) is left PP if and only if it is right PP.

It is natural to ask if some of these properties can be extended from a ring R to the polynomial ring $(R[x], +, \cdot)$ or formal power series ring $(R[[x]], +, \cdot)$ and vice versa. Armendariz [1] and Jøndrup [28] obtained the following results:

Theorem A [1]. *Let R be a reduced ring. Then $(R[x], +, \cdot)$ is a PP ring if and only if R is a PP ring.*

Theorem 1.2 [28]. *Assume R is a commutative ring. Then $(R[x], +, \cdot)$ is PP if and only if $R[x]$ is PP.*

Recall that a ring or nearring is *reduced* if it contains no nonzero nilpotent elements. Armendariz provided an example to show that the reduced condition is not superfluous. In the review of this paper, Burgess [Math. Reviews 51 # 3224] indicated that [1, Theorem A] is false if "$R[x]$" is replaced by "$R[[x]]$". The defects had been fixed by introducing a generalized annihilator conditions [8]. A ring R is called a *right principally quasi-Baer* (or *right* *p.q.-Baer*) ring if the right annihilator of a principal right ideal is generated by an idempotent. Equivalently, R is right p.q.-Baer if R modulo the right annihilator of any principal right ideal is projective. If R is both right and left p.q.-Baer, then R is said to be *p.q.-Baer*. It is shown by Birkenmeier, Kim and Park that:

Theorem 3.1 [8]. *R is a right p.q.-Baer ring if and only if $(R[x], +, \cdot)$ is a right p.q.-Baer ring.*

However, replacing $R[x]$ by $R[[x]]$ is still false in above theorem. Counterexamples had been given in [8, Example 2.3] and [32, Example 4].

Three commonly used operations for polynomials are addition "$+$", multiplication "$\cdot$" and substitution "$\circ$", respectively. Observe that $(R[x], +, \cdot)$ is a ring and $(R[x], +, \circ)$ is a left nearring where the substitution indicates substitution of $f(x)$ into $g(x)$, explicitly $f(x) \circ g(x) = g(f(x))$ for any $f(x), g(x) \in R[x]$. It is natural to investigate the nearring of polynomials $R[x]$, the zero-symmetric nearring of polynomials $R_0[x]$ and the zero-symmetric nearring of formal power series $R_0[[x]]$ when the ring R is equipped with certain annihilator conditions. Motivated by these observations, the author and Birkenmeier [4, 5, 6] initiated the study of various annihilator conditions in the class of nearrings. Let N be a left nearring and S a nonempty subset of N. Denote $r\mathrm{Ann}_N(S) = \{a \in N \mid Sa = 0\}$ and $\ell\mathrm{Ann}_N(S) = \{a \in N \mid aS = 0\}$. If no confusion will arise, the subscript may be omitted. Let $a \in N$ be arbitrary, we define the Rickart-type annihilator conditions [4] in the class of nearrings by describing the following classes:

(1) $N \in \mathcal{R}_{r1}$ if $r\mathrm{Ann}(a) = eN$ for some idempotent $e \in N$;
(2) $N \in \mathcal{R}_{r2}$ if $r\mathrm{Ann}(a) = r\mathrm{Ann}(e)$ for some idempotent $e \in N$;
(3) $N \in \mathcal{R}_{\ell1}$ if $\ell\mathrm{Ann}(a) = Ne$ for some idempotent $e \in N$;
(4) $N \in \mathcal{R}_{\ell2}$ if $\ell\mathrm{Ann}(a) = \ell\mathrm{Ann}(e)$ for some idempotent $e \in N$.

The $\mathcal{R}_{r2}$ condition is actually considered for ring with involution [2, p.28]. If N is a ring with unity then either one of the above four conditions is equivalent to N being right PP or left PP. Conditions (1) and (3) are direct analogue of the ring case but invoking the $\mathcal{R}_{r1}$ condition is severe in a nearring because it requires a right ideal $r\mathrm{Ann}(a)$ to be equal to a right N-subgroup eN. A reduced regular nearring with unity certainly satisfies this requirement but it is false in general.

The extension of Baer rings to the nearring of polynomials $R[x]$, $R_0[x]$ and the nearring of formal power series $R_0[[x]]$ had been investigated in [4, 5, 6]. The extensions of a Rickart ring and its generalizations are barely discussed. The aim of this paper is to fulfil these investigations for the unsolved cases and questions. The formal power series extension of a Baer ring was studied in [5, 9]. Thus it is natural to ask: *What can be said about various Rickart-type annihilator conditions for power series under addition and substitution?* Unfortunately, the substitution of one formal power series into another may have no meaning in general. We use $f(x)$ to denote the formal power series $\sum_{i=0}^{\infty} f_i x^i$ where f_i are in a ring R with unity and $f(x) \circ g(x)$ indicates substitution of $f(x)$ into $g(x)$. We use $R[[x]]$ to denote the set of formal power series over a ring R. Let $f(x) = \sum_{i=0}^{\infty} f_i x^i$ and $(x)g = \sum_{j=0}^{\infty} g_j x^j$. Observe that

$$f(x) \circ g(x) = \left(g_0 + \sum_{i=1}^{\infty} g_i f_0^i \right) + \sum_{p=1}^{\infty} \left(\sum_{j=1}^{\infty} g_j C_p^{(j)} \right) x^p,$$
$$\text{where } C_p^{(j)} = \sum_{u_1+\cdots+u_j=p} f_{u_1} f_{u_2} \cdots f_{u_j} \text{ for } p \in \{1, 2, 3, \cdots\}.$$

A necessary requirement for $f(x) \circ g(x)$ to be well defined is that $g_0 + \sum_{i=1}^{\infty} g_i f_0^i \in R$. One way to solve this problem is to introduce a topology on the ring R. Of particular interest is when R is the complex number field $\mathbb{C}$. The entire function has a unique power series expansion at 0, the substitution is well defined on the power series expansions of entire functions [14].

Another approach is to consider the formal power series with zero constant terms. Note that assuming $f_0 = g_0 = 0$ then $C_p^{(j)} = 0$ when $j > p$ and so the coefficient of each term in the expression of $f(x) \circ g(x)$ will be a finite sum of elements from R. Hence the operation of substitution on these power series is well defined. In the sequel, the collection of all power series with zero constant terms using the operations of addition and substitution is denoted by $R_0[[x]]$ unless specifically indicated otherwise (i.e., $R_0[[x]]$ denotes $(R_0[[x]], +, \circ)$). Observe that the system $(R_0[[x]], +, \circ)$ is a 0-symmetric abelian nearring. Hence the theory of nearrings provides a natural framework in which to study power series under the operations of addition and substitution. For expositions using this approach one may refer to [16, 30, **?**]. Moreover, there has been recent interest by group theorists in pro-p groups and the Nottingham groups [13, 31]. In fact the group studied in [26, 27] is a normal subgroup of the group of units of $(R_0[[x]], +, \circ)$ when R is a commutative ring, and the Nottingham group is a normal subgroup of the group of units of $(R_0[[x]], +, \circ)$ when R is a finite field of characteristic $p > 2$.

Another reason (or motivation?)

Theorem 1. *Let R be an IFP ring with unity. Then the following are equivalent.*

(1) *R is Rickart;*
(2) $(R[x], +, \circ) \in \mathcal{R}_{r2}$*;*
(3) $(R_0[x], +, \circ) \in \mathcal{R}_{r2}$*;*
(4) $(R[x], +, \cdot)$ *is Rickart.*

The following example shows that assuming R an IFP ring is not superfluous.

Example 2. There is a Rickart ring R such that the ring $(R[x], +, \cdot)$ is not Rickart and the nearring $(R_0[x], +, \circ) \notin \mathcal{R}_{r2}$. Let R be the 2-by-2 matrix ring over the ring of integers $\mathbb{Z}$. Then R is a Baer ring (hence a Rickart ring), but $(R[x], +, \cdot)$ is not right (left) Rickart (hence not Rickart) because the right annihilator of $\left(\begin{smallmatrix} 2 & x \\ 0 & 0 \end{smallmatrix}\right)$ and the left annihilator $\left(\begin{smallmatrix} 2 & 0 \\ x & 0 \end{smallmatrix}\right)$ are not generated by an idempotent, respectively. This example (due to P.M. Cohn) is presented in [1, 8, 28].

Next, we will show that the nearring $(R_0[x], +, \circ) \notin \mathcal{R}_{r2}$ (hence $(R[x], +, \circ) \notin \mathcal{R}_{r2}$). Let $\eta(x) = \sum_{i=1}^{m} \eta_i x^i$ be an idempotent in $(R_0[x], +, \circ)$. It is immediate that $\eta_1^2 = \eta_1$ is an idempotent in R. If $\eta(x)$ is a nonzero idempotent, then $\eta_1 \neq 0$. Otherwise, if $\eta(x) = \sum_{i=2}^{m} \eta_i x^i$ then, by comparing coefficients from the equation $\eta(x) \circ \eta(x) = \eta(x)$, it implies that $\eta_2 = 0$. Inductively, it shows $\eta(x) = 0$. Hence $\eta_1 \neq 0$. It is now not difficult to see that the polynomial $x^2 \notin r\mathrm{Ann}(\eta(x))$ since

$$\eta(x) \circ x^2 = \eta(x) \cdot \eta(x) = \eta_1 x^2 + \left(\sum_{\substack{i+j=k \\ 1 \leqslant i,j \leqslant m}} \eta_i \eta_j \right) x^k.$$

Let $f(x) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) x \in R_0[x]$. Then $x^2 \in r\mathrm{Ann}(f(x))$ but $x^2 \notin r\mathrm{Ann}(\eta(x))$ for any idempotent in the nearring $(R_0[x], +, \circ)$. Thus $(R_0[x], +, \circ) \notin \mathcal{R}_{r2}$.

Nearring of formal power series

Lemma 3. *Let R be an IFP ring with unity. If $\eta(x) = \sum_{i=1}^{\infty} \eta_i x^i \in R_0[[x]]$ is an idempotent, then $\eta(x) = \eta_1 x$ with $\eta_1^2 = \eta_1$.*

Proposition 4. *Let R be an IFP ring with unity. If $R_0[[x]] \in \mathcal{R}_{r1} \cup \mathcal{R}_{r2} \cup \mathcal{R}_{\ell 1} \cup \mathcal{R}_{\ell 2}$, then R is a Rickart ring.*

Proposition 5. *Let R be a finite reduced ring with unity. If R is a Rickart ring, then $R_0[[x]] \in \mathcal{R}_{r1} \cup \mathcal{R}_{r2} \cup \mathcal{R}_{\ell 1} \cup \mathcal{R}_{\ell 2}$.*

In the above Propositions, the existence of unity is assumed but ,in fact, it is not quite necessary. The following example shows that assuming R a finite reduced ring is not superfluous.

Example 6. There exists a commutative reduced ring R such that $R \in \mathcal{R}_{r2} \cup \mathcal{R}_{\ell 2}$ but $(R_0[[x]], +, \circ) \notin \mathcal{R}_{r1} \cup \mathcal{R}_{r2} \cup \mathcal{R}_{\ell 1} \cup \mathcal{R}_{\ell 2}$. Indeed, let $R = \bigoplus \mathbb{Z}_2$, a direct sum of infinitely many copies of the field $\mathbb{Z}_2$. Note that each element in R is an idempotent, so $R \in \mathcal{R}_{r2} \cup \mathcal{R}_{\ell 2}$. Let $e_1 = (1, 0, 0, \cdots)$, $e_2 = (0, 1, 0, \cdots)$, $\cdots$, $e_j = (0, \cdots, 0, 1, 0, \cdots)$, $\cdots \in R$, where e_j denotes the element in R with j^{th}-component equals to 1 and 0 elsewhere for all $j \in \mathbb{N}$. Observe that all the monomials ax are idempotents in $R_0[[x]]$ and vice versa. Since $R_0[[x]]$ is reduced [5, Proposition 3.1], and thus all the idempotents are central. It is not difficult to see that $\ell\mathrm{Ann}(ax) = r\mathrm{Ann}(ax) \neq \{0\}$. For instance, if $a = (1, 1, 0, \cdots)$ and $b = (0, 0, 1, 0, \cdots)$, then $bx \in \ell\mathrm{Ann}(ax) = r\mathrm{Ann}(ax)$. Explicitely, if j is the largest number such that the j^{th}-component of $a \in R$ is nonzero, then $e_{j+1}x \in \ell\mathrm{Ann}(ax) = r\mathrm{Ann}(ax)$.

Let $f(x) = \sum_{i=1}^{\infty} e_i x^i \in R_0[[x]]$. If $g(x) = \sum_{j=1}^{\infty} g_j x^j \in r\mathrm{Ann}(f(x))$, then $g_j e_i = 0$ for all $i, j \in \mathbb{N}$ by [5, Lemm 3.3] and thus $g_j = 0$ for all $j \in \mathbb{N}$. Therefore $r\mathrm{Ann}(f(x)) = 0$. Similarly, $\ell\mathrm{Ann}(f(x)) = 0$ and thus $(R_0[[x]], +, \circ) \notin \mathcal{R}_{r2} \cup \mathcal{R}_{\ell 2}$.

On the other hand, let $h(x) = \sum_{i=1}^{\infty} e_{2i} x^{2i}$. Then $e_{2i-1}x \in r\mathrm{Ann}(h(x)) \cap \ell\mathrm{Ann}(h(x))$ for all $i \in \mathbb{N}$. Since aR is finite and $R_0[[x]] \circ (ax) = (ax) \circ R_0[[x]] = (aR)_0[[x]]$, it follows that $r\mathrm{Ann}(h(x)) \neq (ax) \circ R_0[[x]]$ and $\ell\mathrm{Ann}(h(x)) \neq R_0[[x]] \circ (ax)$ for all $a \in R$. Thus $(R_0[[x]], +, \circ) \notin \mathcal{R}_{r1} \cup \mathcal{R}_{\ell 1}$.

Theorem 7. *Let R be a finite reduced ring with unity. Then the following are equivalent.*

(1) *R is Rickart;*
(2) $(R[x], +, \circ) \in \mathcal{R}_{r2}$*;*
(3) $(R_0[x], +, \circ) \in \mathcal{R}_{r2}$*;*
(4) $(R_0[[x]], +, \circ) \in \mathcal{R}_{r2}$*;*
(5) $(R[x], +, \cdot)$ *is Rickart.*

The following example shows that assuming R a finite ring in above result is not superfluous.

Example 8. There exists a reduced ring T with unity such that T is a Rickart ring but $(T_0[[x]], +, \circ) \notin \mathcal{R}_{r2}$. Let $R = \prod \mathbb{Z}_2 = \{a \colon \mathbb{N} \to \mathbb{Z}_2\}$ be the ring of direct product of the field $\mathbb{Z}_2$. Let $T = \{a \in R \mid a(i)$ is eventually constant for $i \in \mathbb{N}\}$. Then T is a proper subring of R. Observe that T is a reduced ring with unity 1. Since each element in T is an idempotent, it is a Rickart ring. In fact, $r\mathrm{Ann}_T(a) = (1-a)T$. Let $e_i \in T$ such that

$$e_i(j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j; \end{cases}$$

and $f(x) = \sum_{i=1}^{\infty} e_{2i}x^{2i}$. Observe that $e_s e_t = 0$ if $s \neq t \in \mathbb{N}$. Then $g(x) = \sum_{i=1}^{\infty} e_{2i-1}x^{2i-1} \in r\mathrm{Ann}(f(x)) = \ell\mathrm{Ann}(f(x))$ by [5, Lemma 3.3]. By Lemma 4.1, idempotents in $T_0[[x]]$ are ax for all $a \in T$ and

$$r\mathrm{Ann}(ax) = \{\sum_{i=1}^{\infty} h_i x^i \mid h_i \in (1-a)T\}.$$

It is immediate that $g(x) \notin r\mathrm{Ann}(ax)$ for all $a \in T$. Thus $T_0[[x]] \notin \mathcal{R}_{r2}$.

Formal power series ring

It is known that even reduced Rickart ring is not stable when extending to the formal power series ring $(R[[x]], +, \cdot)$. In fact, counterexamples had been provided to show that $(R[[x]], +, \cdot)$ is not Rickart when R is a commutative von Neumann regular ring [8, Example 2.3] and [32, Example 4]. However, Theorem 4.5 did motivate the following question: *Is it true that the formal power series ring $(R[[x]], +, \cdot)$ a Rickart ring when R is a finite reduced Rickart ring?* The answer is affirmative as presented in the following discussions. Be aware that in the following we are studying the formal power series ring $(R[[x]], +, \cdot)$ instead of the the formal power series nearring $(R_0[[x]], +, \circ)$.

Lemma 9. *Let R be an IFP ring with unity. If $\eta(x)$ is an idempotent in the formal power series ring $(R[[x]], +, \cdot)$, then $\eta(x) = e$ is a constant polynomial where e is an idempotent in the ring R.*

Lemma 10. *Let R be a reduced ring and $f(x) = \sum_{i=0}^{\infty} f_i x^i$, $g(x) = \sum_{j=0}^{\infty} g_j x^j \in (R[[x]], +, \cdot)$. Then $f(x) \cdot g(x) = 0$ if and only if $f_i g_j = 0$ for all $i, j \in \mathbb{N} \cup \{0\}$.*

Theorem 11. *Let R be a finite reduced ring with unity. Then R is Rickart if and only if the formal power series ring $(R[[x]], +, \cdot)$ is Rickart.*

The following example shows that assuming R finite is not superfluous.

Example 12. There is a commutative reduced ring R with unity such that R is a Rickart ring but the formal power series ring $(R[[x]], +, \cdot)$ is not Rickart. Let F be a field and let

$$R = \{(a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} F_n \mid a_n \text{ is eventually constant}\},$$

a subring of $\prod_{n=1}^{\infty} F_n$ where $F_n = F$ for all $n \in \mathbb{N}$. This ring R is a commutative von Neumann regular ring (hence a reduced Rickart ring) but the ring $(R[[x]], +, \cdot)$ is not Rickart [8, Example 2.3]. See also [32, Example 4] for a commutative Rickart ring R but $(R[[x]], +, \cdot)$ is not Rickart.

2nd. approach

Throughout this note, R denotes a ring with unity. Recall that R is called a (*quasi-*)*Baer* ring if the right annihilator of every (right ideal) nonempty subset of R is generated, as a right ideal, by an idempotent of R. Baer rings are introduced by Kaplansky [29] to abstract various properties of AW^*-algebras and von Neumann algebras. Quasi-Baer rings, introduced by Clark [15], are used to characterize when a finite dimensional algebra over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. The definition of a (quasi-) Baer ring is left-right symmetric [15, 29].

In [10], Birkenmeier, Kim and Park initiated the study of right principally quasi-Baer rings. A ring R is called *right principally quasi-Baer* (or simply *right p.q.-Baer*) if the right annihilator of a principal right ideal is generated, as a right ideal, by an idempotent. Equivalently, R is right p.q.-Baer if R modulo the right annihilator of any principal right ideal is projective. If R is both right and left p.q.-Baer, then it is called *p.q.-Baer*. The class of p.q.-Baer rings include all biregular rings, all quasi-Baer rings and all abelian PP rings. See [10] for more details.

Ore extensions or polynomial extensions of (quasi-)Baer rings and their generalizations are extensively studied recently ([4] to [11] and [22] to [25]). It is proved in [9, Theorem 1.8] that a ring R is quasi-Baer if and only if $R[[X]]$ is quasi-Baer, where X is an arbitrary nonempty set of not necessarily commuting indeterminates. In [8, Theorem 2.1], it is shown that R is right p.q.-Baer if and only if $R[x]$ is right p.q.-Baer. But it is not equivalent to that $R[[x]]$ is right p.q.-Baer. In fact, there exists a commutative von Neumann regular ring R (hence p.q.-Baer) such that the ring $R[[x]]$ is not p.q.-Baer [8, Example 2.6]. In [33, Theorem 3], a necessary and sufficient condition for semiprime ring under which the ring $R[[x]]$ is right p.q.-Baer are given. It is shown that $R[[x]]$ is right p.q.-Baer if and only if R is right p.q.-Baer and any countable family of idempotents in R has a generalized join when all left semicentral idempotents are central. Indeed, for a right p.q.-Baer ring, asking the set of left semicentral idempotents $\mathcal{S}_\ell(R)$ equals to the set of central idempotents $B(R)$ is equivalent to assume R is semiprime [10, Proposition 1.17]. In this note, the condition requiring all left semicentral idempotents being central is shown to be redundant. We show that: *The ring $R[[x]]$ is right p.q.-Baer if and only if R is p.q.-Baer and every countable subset of right semicentral idempotents has a generalized countable join.* This theorem properly generalizes Fraser and Nicholson's result in the class of reduced PP rings [19, Theorem 3] and Liu's result in the class of semiprime p.q.-Baer rings [33, Theorem 3]. For simplicity of notations, denote $\mathbb{N} = \{0, 1, 2, \cdots\}$ be the set of natural numbers.

Annihilators and left semicentral idempotents

Lemma 13. *Let* $f(x) = \sum_{i=0}^{\infty} f_i x^i$, $g(x) = \sum_{j=0}^{\infty} g_j x^j \in R[[x]]$. *Then the following are equivalent.*

(1) $f(x)R[[x]]g(x) = 0$;
(2) $f(x)Rg(x) = 0$;
(3) $\sum_{i+j=k} f_i a g_j = 0$ *for all* $k \in \mathbb{N}$, $a \in R$.

Recall that an idempotent $e \in R$ is called *left* (resp. *right*) *semicentral* [3] if $re = ere$ (resp. $er = ere$) for all $r \in R$. Equivalently, $e = e^2 \in R$ is left (resp. right) semicentral if eR (resp. Re) is an ideal of R. Since the right annihilator of a right ideal is an ideal, we see that the right annihilator of a right ideal is generated by a left semicentral idempotents in a right p.q.-Baer ring. The set of left (resp. right) semicentral idempotents of R is denoted $\mathcal{S}_\ell(R)$ (resp. $\mathcal{S}_r(R)$). The following result is used frequently later in this note.

Lemma 14. *[10, Lemma 1.1] Let* e *be an idempotent in a ring* R *with unity. Then the following conditions are equivalent.*

(1) $e \in \mathcal{S}_\ell(R)$;
(2) $1 - e \in \mathcal{S}_r(R)$;
(3) $(1-e)Re = 0$;
(4) eR *is an ideal of* R;
(5) $R(1-e)$ *is an ideal of* R.

To prove the main result, we first characterize the left semicentral idempotents in $R[[x]]$.

Proposition 15. *Let* $\varepsilon(x) = \sum_{i=0}^{\infty} \varepsilon_i x^i \in R[[x]]$. *Then* $\varepsilon(x) \in \mathcal{S}_\ell(R[[x]])$ *if and only if*

(1) $\varepsilon_0 \in \mathcal{S}_\ell(R)$;
(2) $\varepsilon_0 r \varepsilon_i = r\varepsilon_i$ *and* $\varepsilon_i r \varepsilon_0 = 0$ *for all* $r \in R$, $i = 1, 2, \cdots$;
(3) $\sum_{\substack{i+j=k \\ i,j \geqslant 1}} \varepsilon_i r \varepsilon_j = 0$ *for all* $r \in R$ *and* $k \geqslant 2$.

Corollary 16. *[8, Proposition 2.4(iv)] Let* R *be a ring with unity and* $\varepsilon(x) = \sum_{i=0}^{\infty} \varepsilon_i x^i \in \mathcal{S}_\ell(R[[x]])$. *Then* $\varepsilon(x)R[[x]] = \varepsilon_0 R[[x]]$.

Generalized countable join

Let R be a ring with unity and $E = \{e_0, e_1, e_2, \cdots\}$ a countable subset of $\mathcal{S}_r(R)$. We say E has a *generalized countable join* e if, given $a \in R$, there exists $e \in \mathcal{S}_r(R)$ such that

(1) $e_i e = e_i$ for all $i \in \mathbb{N}$;

(2) if $e_i a = e_i$ for all $i \in \mathbb{N}$, then $ea = e$.

Note that if there exists an element $e \in R$ satisfies conditions (1) and (2) above, then $e \in \mathcal{S}_r(R)$. Indeed, the condition (1): $e_i e = e_i$ for all $i \in \mathbb{N}$ implies $ee = e$ by (2) and so e is an idempotent. Further, let $a \in R$ be arbitrary. Then the element $d = e - ea + eae$ is an idempotent in R and $e_i d = e_i$ for all $i \in \mathbb{N}$. Thus $ed = e$ by (2). Note that $ed = e(e - ea + eae) = d$. Consequently, $e = d = e - ea + eae$ or $ea = eae$. Thus $e \in \mathcal{S}_r(R)$.

Note that a generalized countable join e, if it exists, is indeed a join if $\mathcal{S}_r(R)$ is a lattice. Recall that when R is an abelian ring (i.e., every idempotent is central), then the set $B(R) = \mathcal{S}_r(R)$ of all idempotents in R is a Boolean algebra where $e \leqslant d$ means $ed = e$. Let e be a join of $E = \{e_0, e_1, e_2, \cdots\}$ in $B(R)$ where R is a reduced PP ring. That is e satisfies (1) $e_i e = e_i$ for all $i \in \mathbb{N}$; $(2')$ if $e_i d = e_i$ for all $i \in \mathbb{N}$ and any $d \in B(R)$, then $ed = e$. Given an arbitrary $a \in R$, then $1 - a = pu$ for some central idempotent $p \in R$ and some $u \in R$ such that $r\mathrm{Ann}_R(u) = 0 = \ell\mathrm{Ann}_R(u)$ [19, Proposition 2]. Observe that if $e_i a = e_i$ for all $i \in \mathbb{N}$, then $e_i(1-a) = e_i pu = 0$. It follows that $e_i p = 0$ for all $i \in \mathbb{N}$ since $\ell\mathrm{Ann}_R(u) = 0$. Thus $ep = 0$ or $e(1-a) = epu = 0$. Therefore $ea = e$ and e is a generalized countable join of E. In other words, a generalized countable join is a join and vice versa in the class of reduced PP rings.

Be aware that $(\mathcal{S}_r(R), \leqslant)$ is not partially ordered by defining $d \leqslant e$ when $de = d$ in an arbitrary ring R. This relation is reflexive, transitive but not antisymmetric. However, let $a, b \in \mathcal{S}_r(R)$ and define $a \sim b$ if $a = ab$ and $b = ba$. Then $\sim$ is an equivalence relation on $\mathcal{S}_r(R)$ and $(\mathcal{S}_r(R)/\sim, \leqslant)$ is a partially ordered set. In the case when $(\mathcal{S}_r(R)/\sim, \leqslant)$ is a complete lattice, then a generalized countable join exists for any subset of $\mathcal{S}_r(R)$. In particular when R is a Boolean ring or a reduced PP ring, then the generalized countable join is indeed a join in R.

In [33, Definition 2], Liu defined the notion of generalized join for a countable set of idempotents. Explicitely, let $\{e_0, e_1, \cdots\}$ be a countable family of idempotents of R. The set $\{e_0, e_1, \cdots\}$ is said to have a *generalized join* e if there exists $e = e^2$ such that

(i) $e_i R(1-e) = 0$;

(ii) if d is an idempotent and $e_i R(1-d) = 0$ then $eR(1-d) = 0$.

Observe that

$$e_i r(1-e) = e_i r e_i (1-e) = e_i r(e_i - e_i e),$$

when $e_i \in \mathcal{S}_r(R)$. Thus $e_i = e_ie$ if and only if $e_ir(1-e) = 0$ for all $r \in R$ when $e_i \in \mathcal{S}_r(R)$ for all $i \in \mathbb{N}$. Now, let $E = \{e_0, e_1, e_2, \cdots\} \subseteq \mathcal{S}_r(R)$ and e a generalized countable join of E. To show e is a generalized join (in the sense of Liu), it remains to show condition (ii) holds. Let f be an idempotent in R such that $e_iR(1-f) = 0$. Then, in particular, $e_i(1-f) = 0$ for all $i \in \mathbb{N}$. Thus $e(1-f) = 0$ by hypothesis. It follows that $er(1-f) = ere(1-f) = 0$ and thus $eR(1-f) = 0$. Therefore, e is a generalized join of E. Thus, in the content of right semicentral idempotents, a generalized countable join is a generalized join in the sense of Liu.

Conversely, let $e \in \mathcal{S}_r(R)$ be a generalized join (in the sense of Liu) of the set $E = \{e_0, e_1, e_2, \cdots\} \subseteq \mathcal{S}_r(R)$. Observe that condition (ii) is equivalent to

(ii') if d is an idempotent and $e_id = e_i$ then $ed = e$.

Let $a \in R$ be arbitrary such that $e_ia = e_i$ for all $i \in \mathbb{N}$. Then condition (ii') and a similar argument used in the case of reduced PP rings implies that $ea = e$. Thus e is a generalized countable join. Therefore, in the content of right semicentral idempotents, Liu's generalized join is equivalent to generalized countable join.

Main Result

Theorem 17. *Let R be a ring with unity. Then $R[[x]]$ is right p.q.-Baer if and only if R is right p.q.-Baer and every countable subset of $\mathcal{S}_r(R)$ has a generalized countable join.*

Since Liu's generalized join is equivalent to generalized countable join in the set of right semicentral idempotents $\mathcal{S}_r(R)$. The following result is immediated from Theorem 5.

Corollary 18. *[33, Theorem 3] Let R be a ring such that $\mathcal{S}_\ell(R) \subseteq B(R)$. Then $R[[x]]$ is right p.q.-Baer if and only if R is right p.q.-Baer and any countable family of idempotents in R has a generalized join.*

Corollary 19. *[19, Theorem 3] If R is a ring then $R[[x]]$ is a reduced PP ring if and only if R is a reduced PP ring and any countable family of idempotents in R has a join in $B(R)$.*

3rd. Approach

Introduction

Recall from [29] that R is a *Baer* ring if R has a unity and the right annihilator of every nonempty subset of R is generated, as a right ideal, by an idempotent. The study of Baer rings has its roots in functional analysis ([2] and [29]). For example, every von Neumann algebra (e.g., the algebra of all bounded linear operators on a Hilbert space) is a Baer ring. Kaplansky shows in [29] that the definition of a Baer ring is left-right symmetric.

In [4], various annihilator conditions on polynomials under addition and substitution are investigated. The formal power series extension of a Baer ring was studied in [9]. Thus it is natural to ask: *What can be said about various Baer-type annihilator conditions for power series under addition and substitution?* Unfortunately, the substitution of one formal power series into another may have no meaning in general. We use $(x)f$ to denote the formal power series $\sum_{i=0}^{\infty} f_i x^i$ where f_i are in a ring R with unity and $(x)f \circ (x)g$ indicates substitution of $(x)f$ into $(x)g$. Composition of functions is performed in a similar manner. We use $R[[x]]$ to denote the set of formal power series over a ring R. Let $(x)f = \sum_{i=0}^{\infty} f_i x^i$ and $(x)g = \sum_{j=0}^{\infty} g_j x^j$. Through a lengthy calculation, we have

$$(x)f \circ (x)g = ((x)f)g = \left(g_0 + \sum_{i=1}^{\infty} g_i f_0^i \right) + \sum_{p=1}^{\infty} \left(\sum_{j=1}^{\infty} g_j c_p^{(j)} \right) x^p,$$

$$\text{where } c_p^{(j)} = \sum_{u_1+\cdots+u_j=p} f_{u_1} f_{u_2} \cdots f_{u_j} \text{ for } p \in \{1,2,3,\cdots\}.$$

A necessary requirement for $(x)f \circ (x)g$ to be well defined is that $g_0 + \sum_{i=1}^{\infty} g_i f_0^i \in R$. One way to solve this problem is to introduce a topology on the ring R. Of particular interest is when R is the complex number field $\mathbb{C}$. Recall a function $f : D \to \mathbb{C}$ is called *analytic* on an open set $D \subseteq \mathbb{C}$ if f has a power series expansion at every point in D. It is called *entire* if f is analytic on $\mathbb{C}$. Since the composition of two entire functions is entire [14, Proposition 5.1, p.22] and each entire function has a unique power series expansion at 0, the substitution is well defined on the power series expansions of entire functions. This was actually done by Cartan [14] in the early 60's. In what follows, $\mathcal{E}(\mathbb{C})$ denotes the nearring of entire functions $(\mathcal{E}(\mathbb{C}), +, \circ)$.

Another approach is to simply consider the formal power series with positive orders (i.e., with zero constant term). Note that assuming $f_0 = g_0 = 0$ then $c_p^{(j)} = 0$ when $j > p$ and so the coefficient of each term in the expression of $(x)f \circ (x)g$ will be a finite sum of elements from R. Hence the operation of substitution on these power series is well defined. In the sequel, the collection of all power series with positive orders using the operations of addition and substitution is denoted by $R_0[[x]]$ unless specifically indicated otherwise (i.e., $R_0[[x]]$ denotes $(R_0[[x]], +, \circ)$). Observe that the system $(R_0[[x]], +, \circ)$ is a 0-symmetric left nearring. Hence the theory of nearrings provides the natural framework in which to study power series under the operations of addition and substitution. For expositions using this approach one may refer to Clay [16], Kautschitsch and Mlitz [30]. Moreover, there has been recent interest by group theorists in pro-p groups and the Nottingham groups [**?**, **?**]. In fact the group studied in [**?**, **?**] is a normal subgroup of the group of units of $(R_0[[x]], +, \circ)$ when R is a commutative ring, and the Nottingham group is a normal subgroup of the group of units of $(R_0[[x]], +, \circ)$ when R is a finite field of characteristic $p > 2$. Note that power series without constant terms often arise as a unique formal solution to the differential equation $\frac{dy}{dx} = f$ where f is an analytic function over $\mathbb{C}^2$ [14, pp.210–212].

Throughout this paper all rings are associative and all nearrings are left nearrings. We use R and N to denote a ring and nearring, respectively. For a nonempty $S \subseteq N$, $r_N(S) = \{a \in N \mid Sa = 0\}$ and $\ell_N(S) = \{a \in N \mid aS = 0\}$. If the context is clear, the subscript may be omitted. *Baer-type annihilator conditions* in the class of nearrings have been defined in [**?**] as the follows:
(1) $N \in \mathcal{B}_{r1}$ if the right annihilator $r(S) = eN$ for some idempotent $e \in N$;
(2) $N \in \mathcal{B}_{r2}$ if the right annihilator $r(S) = r(e)$ for some idempotent $e \in N$;
(3) $N \in \mathcal{B}_{\ell 1}$ if the left annihilator $\ell(S) = Ne$ for some idempotent $e \in N$;
(4) $N \in \mathcal{B}_{\ell 2}$ if the left annihilator $\ell(S) = \ell(e)$ for some idempotent $e \in N$.

When S is a singleton, the *Rickart-type annihilator conditions* on nearrings are also defined and denoted similarly except $\mathcal{B}$ is replaced by $\mathcal{R}$. In [2, p.28], the $\mathcal{R}_{r2}$ condition is considered for rings with involution. If N is a ring with unity then $N \in \mathcal{B}_{r1} \cup \mathcal{B}_{r2} \cup \mathcal{B}_{\ell 1} \cup \mathcal{B}_{\ell 2}$ is equivalent to N being a Baer ring.

Analytical approach to formal power series

The ring of analytic functions $(\mathcal{E}(D), +, \cdot)$ over a connected open set D is an integral domain [14, Corollary 1, p.40]. However this does not apply to the nearring $\mathcal{E}(\mathbb{C})$ as shown in the following proposition.

Proposition 20. *Let $f, g \in \mathcal{E}(\mathbb{C})$ be two entire functions where $g \neq 0$. If $f \circ g = 0$, then f is a constant.*

Corollary 21. *The idempotents in $\mathcal{E}(\mathbb{C})$ are either constants or the identity function μ.*

Theorem 22. *Let $\mathcal{E}(\mathbb{C})$ be the nearring of entire functions. Then $\mathcal{E}(\mathbb{C}) \in \mathcal{R}_{r2}$ but not in $\mathcal{B}_{r2}$.*

A nearring N is called *integral* if N has no nontrivial zero divisors. We have seen the nearring $\mathcal{E}(\mathbb{C})$ is not integral. However, its subnearring $\mathcal{E}_0(\mathbb{C}) = \{f \in \mathcal{E}(\mathbb{C}) \mid (0)f = 0\}$ is indeed integral, as a consequence of Proposition 2.1. Thus $\mathcal{E}_0(\mathbb{C})$ satisfies all the Baer-type annihilator conditions as expected by [**?**, Proposition 1.8]. We write these observations in the following result.

Proposition 23. *The 0-symmetric nearring $\mathcal{E}_0(\mathbb{C})$ is integral with unity μ, and $\mathcal{E}_0(\mathbb{C}) \in \mathcal{B}_{r1} \cap \mathcal{B}_{r2} \cap \mathcal{B}_{\ell 1} \cap \mathcal{B}_{\ell 2}$.*

Algebraic approach to formal power series

We now study Baer-type annihilator conditions on formal power series via the algebraic approach. Throughout this section, $R_0[[x]]$ will denote the nearring of formal power series $(R_0[[x]], +, \circ)$ with positive orders. If $S \subseteq R_0[[x]]$ then $r(S) = \{f \in R_0[[x]] \mid S \circ f = 0\}$ and $\ell(S) = \{f \in R_0[[x]] \mid f \circ S = 0\}$.

Proposition 24. *Let R be a ring. Then R is reduced if and only if $R_0[[x]]$ is reduced.*

Lemma 25. *Let R be a ring. If $(x)\eta = \sum_{i=1}^{\infty} \eta_i x^i \in R_0[[x]]$ is an idempotent, then $\eta_1^2 = \eta_1$. If R is reduced, then $(x)\eta = \eta_1 x$.*

Lemma 26. *Let R be a reduced ring and $(x)f, (x)g \in R_0[[x]]$ with $(x)f = \sum_{i=1}^{\infty} f_i x^i$ and $(x)g = \sum_{j=1}^{\infty} g_j x^j$. Then $(x)f \circ (x)g = 0$ if and only if $g_j f_i = 0$ for all $i, j \in \{1, 2, 3, \cdots\}$.*

If $(x)f = \sum_{i=1}^{\infty} f_i x^i \in R_0[[x]]$, let $S_f^* := \{f_i \mid i \in \mathbb{N}\}$.

Proposition 27. *Let R be a reduced ring. Then*
(1) *$R \in \mathcal{B}_{r1}$ if and only if $R_0[[x]] \in \mathcal{B}_{\ell 1}$;*
(2) *$R \in \mathcal{B}_{r2}$ if and only if $R_0[[x]] \in \mathcal{B}_{\ell 2}$.*

Theorem 28. *Let R be a reduced ring.*
(1) *If R is Baer, then $R_0[[x]] \in \mathcal{B}_{r1} \cap \mathcal{B}_{r2} \cap \mathcal{B}_{\ell 1} \cap \mathcal{B}_{\ell 2}$.*
(2) *If $R_0[[x]] \in \mathcal{B}_{r1} \cup \mathcal{B}_{r2} \cup \mathcal{B}_{\ell 1} \cup \mathcal{B}_{\ell 2}$, then R is Baer.*

Corollary 29. *Let R be a reduced ring. The following are equivalent:*
(1) *R is Baer ;*
(2) *$(R[[x]], +, \cdot)$ is Baer ;*
(3) *$(R_0[[x]], +, \circ) \in \mathcal{B}_{r1} \cup \mathcal{B}_{r2} \cup \mathcal{B}_{\ell 1} \cup \mathcal{B}_{\ell 2}$;*
(4) *$(R_0[[x]], +, \circ) \in \mathcal{B}_{r1} \cap \mathcal{B}_{r2} \cap \mathcal{B}_{\ell 1} \cap \mathcal{B}_{\ell 2}$.*

Proposition 30. *Assume R is a reduced ring. Let S be the subnearring of $R_0[[x]]$ generated by the set $\{ex \mid e = e^2 \in R\}$ and T a subnearring of $R_0[[x]]$. If $R_0[[x]] \in \mathcal{B}_{\nu i}$, where $\nu \in \{r, \ell\}$ and $i \in \{1, 2\}$, and $S \subseteq T$, then $T \in \mathcal{B}_{\nu i}$.*

Example 31. Using Proposition 3.4, the following nearrings satisfy all the Baer-type annihilator conditions discussed in this paper when R is a reduced Baer ring. (i) $\{ax \mid a \in R\}$; (ii) $\{(x)f = \sum_{i=1}^{\infty} a_{2i-1} x^{2i-1} \in R_0[[x]] \mid a_{2i-1} \in R \text{ for all } i \in \mathbb{N}\}$; (iii) $E_0[[x]]$, where E is a subring containing all idempotents of R.

Corollary 32. *The nearring of 0–preserving entire functions $\mathcal{E}_0(\mathbb{C}) \in \mathcal{B}_{r1} \cap \mathcal{B}_{r2} \cap \mathcal{B}_{\ell 1} \cap \mathcal{B}_{\ell 2}$.*

Since there is both a ring and nearring structure on $R_0[[x]]$, it is natural to ask: What are the connections between the ring structure $(R_0[[x]], +, \cdot)$ and the nearring structure $(R_0[[x]], +, \circ)$? Our remaining results address this question.

Let N be a nearring and $0 \subseteq X \subseteq Y \subseteq N$. We say X is *2-essential* in Y, denoted by $X \leq_2^{ess} Y$, if for each nonzero N-subgroup I, $I \subseteq Y$ implies $X \cap I \neq 0$. We use $X \lhd N$ to denote that X is an ideal of N. From [1, Lemma 1] and Lemma 3.3, if $(x)g$ right (left) annihilates $(x)f$ in $(R_0[[x]], +, \cdot)$ then $(x)g$ right (left) annihilates $(x)f$ in $(R_0[[x]], +, \circ)$ when R is reduced. Moreover, the following result shows that if R is a reduced Baer ring then every nearring ideal of $R_0[[x]]$ is 2-essential in a nearring direct summand which is also a ring direct summand. Note that if e is a central idempotent in a ring R, then $ex \circ R_0[[x]] \lhd (R_0[[x]], +, \circ)$ and $ex \circ R_0[[x]] = e \cdot R_0[[x]] \lhd (R_0[[x]], +, \cdot)$. Moreover, all idempotents in a reduced ring or nearring with unity are central.

Theorem 33. *Let R be a reduced Baer ring and $0 \neq I \lhd (R_0[[x]], +, \circ)$. Then there exists $e = e^2 \in R$ such that $I \leq_2^{ess} ex \circ R_0[[x]] \lhd (R_0[[x]], +, \circ)$, and $I \cap B \neq 0$ for every $0 \neq B \lhd (R_0[[x]], +, \cdot)$ such that $R \cdot B \subseteq B$ and $B \subseteq e \cdot R_0[[x]]$. Moreover, if $I \lhd (R_0[[x]], +, \cdot)$, then I is essential as a ring right ideal in the ring direct summand $e \cdot R_0[[x]]$ of the ring $(R_0[[x]], +, \cdot)$.*

Corollary 34. *Let R be a reduced Baer ring.*

(1) *If ϱ is a radical map, then $R_0[[x]] = A \oplus S$ (nearring direct sum), where $\varrho(R_0[[x]]) \leq_2^{ess} A$ and S is ϱ-semisimple.*

(2) *If M is a maximal ideal of $R_0[[x]]$, then $M \leq_2^{ess} R_0[[x]]$.*

Recall $(R, +, \cdot, \circ)$ is called a *composition ring* [16, 30] if $(R, +, \cdot)$ is a ring and $(R, +, \circ)$ is a left nearring satisfying $a \circ (b \cdot c) = (a \circ b) \cdot (a \circ c)$ for all $a, b, c \in R$. Thus $(R_0[[x]], +, \cdot, \circ)$ is a composition ring if R is a commutative ring. With the following definitions, Theorem 3.10 can be applied to the case when $R_0[[x]]$ is a composition ring. A subset I of R is called a *full ideal* of R if I is both a ring ideal and a nearring ideal of R. We call a nonzero full ideal I of a composition ring $(R, +, \cdot, \circ)$ *right 2-essential* in a subcomposition ring T of R, if I has nonzero intersection with every nonzero right ideal of $(R, +, \cdot)$ which is contained in T and I has nonzero intersection with every nonzero right R-subgroup of $(R, +, \circ)$ which is contained in T. Observe that a commutative Baer ring is a reduced ring.

Corollary 35. *Let R be a commutative Baer ring. Then every nonzero full ideal I of the composition ring $(R_0[[x]], +, \cdot, \circ)$ is right 2-essential in the full ideal $e \cdot R_0[[x]]$ for some idempotent $e \in R$.*

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