Additive Set of Idempotents in Rings

Juncheol Han

Pusan National University, Korea

[Jointworked with Sangwon Park]

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1. Introduction and basic definitions

Some Notations

- $R :=$ a ring with idemtity 1
- $I(R) :=$ the set of all nonunits idempotents in R
- $M(R) :=$ the set of all primitive idempotents and 0 in R

Known Results

$\diamond$ D. Dolžan, Multiplicative sets of idempotents in a finite ring, J. Algebra, 2006.

Definition :

1. Let $\leq_1$ denote the usual relation on $I(R)$ defined by
$$e \leq_1 f \Leftrightarrow ef = fe = e.$$

2. An idempotent e is said to be preserves $G(R)$ (the group of all units in R), if the set $eGe \subseteq G(eRe)$.

Theorem 1. *If every minimal idempotent preserves G, then R is a direct sum of local rings and the number of summands equals the maximal number of mutually orthogonal minimal idempotents in R.*

Theorem 2. *If M is closed under multiplicative, then every minimal idempotent preserves G.*

Corollary 3. *Let M be the set of all nonzero minimal idempotents according to $\leq_1$. Then M is closed under multiplication if and only if R is a direct sum of local rings.*

◇ H. K. Grover, D. Khurana and S. Singh, Rings with multiplicative sets of primitive idempotents, Comm. Algebra, 2009.

<u>*Definition* :</u>

1. A ring R is called *connected* if it has no idempotents other than 0 and 1.

2. Two idempotents $e, f \in R$ are said to be *orthogonal* if $ef = fe = 0$.

Theorem 1. *A ring R is a finite direct product of connected rings if and only if $M(R)$ is multiplicative and R has a complete finite set of primitive orthogonal idempotents.*

Theorem 2. *If $M(R)$ is multiplicative, then for any $0 \neq e \in M(R)$ and $u \in G(R)$, $eue \in G(eRe)$ with $(eue)^{-1} = eu^{-1}e$.*

<u>$Definition:$</u>

1. $I(R)$ is said to be *additive* if for all $e, f \in I(R)$ $(e \neq f)$, $e + f \in I(R)$ (equivalently, $ef = -fe$).

Example: Boolean ring

2. $M(R)$ is said to be *additive in* $I(R)$ if for all $e, f \in M(R)$ $(e \neq f)$, $e + f \in I(R)$.

Examples: (1) Boolean ring

(2) A direct product of local rings

Note

1. $I(R)$ is additive $\Rightarrow M(R)$ is additive in $I(R)$.

2. $I(R)$ is additive $\not\Leftarrow M(R)$ is additive in $I(R)$

Example: A finite direct product of infinite fields.

2. Some properties of a ring with additive idempotents

Lemma 2.1. *Let R be a ring. If $I(R)$ is additive, then for all $e, f \in I(R)$, $ef = fe$, i.e., $I(R)$ is commuting.*

Note

(1) $I(R)$ is additive, $\Rightarrow I(R) \subseteq Z(R)$.

(2) $I(R)$ is additive, $\not\Leftarrow I(R) \subseteq Z(R)$.

Example: $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$.

Corollary 2.2. *Let R be a ring. If $I(R)$ is additive, then for all $e, f \in I(R)$ $(e \neq f)$, $2ef = 0$.*

Theorem 2.3. *Let R be a ring. Then $I(R)$ is additive if and only if $I(R)$ is commuting and char(R) = 2.*

Remark 1. Note that Theorem 2.3 exhibits that if R is a ring such that $I(R)$ is additive, then $1 + e \in I(R)$ for all $0 \neq e \in I(R)$.

Lemma 2.4. *Let R be a ring. If $M(R)$ is additive in $I(R)$, then for all $e, f \in M(R)$ $(e \neq f)$, $ef = fe$, and also $M(R) \subseteq Z(R)$.*

Theorem 2.5. *Let R be a ring. If $M(R)$ is additive in $I(R)$, then for all $e, f \in M(R)$ $(e \neq f)$, $ef = fe = 0$.*

Corollary 2.6. *Let R be a ring. Then $M(R)$ is additive in $I(R)$ if and only if $M(R)$ is the set of primitive pairwise orthogonal idempotents.*

Remark 2. Let R be a ring such that $M(R)$ is additive in $I(R)$. Observe that (1) if $eR = fR$ for some $e, f \in M(R)$ ($e, f \neq 0$), then $e = f$; (2) if $e_1, e_2, \cdots, e_n \in M(R)$ are distinct, then $e_1R + e_2R + \cdots + e_nR = e_1R \oplus e_2R \oplus \cdots \oplus e_nR$ with $e_iR \cap e_jR = \{0\}$ for all $i, j = 1, \cdots, n$ ($i \neq j$), and $(e_1 + e_2 + \cdots + e_n)R = e_1R \oplus e_2R \oplus \cdots \oplus e_nR$.

Lemma 2.7. *Let R be a ring such that $M(R)$ is commuting, and let $N \subseteq J(R)$ be an ideal of R. If $\bar{e} = \bar{f} \in R/N$ for some $e, f \in M(R)$, then $e = f$.*

Corollary 2.8. *Let $N \subseteq J(R)$ be an ideal of a ring R such that idempotents in R/N can be lifted to R. If $M(R)$ is commuting, then $|M(R)| = |M(R/N)|$.*

Theorem 2.9. *Let $N \subseteq J(R)$ be an ideal of R such that idempotents in R/N can be lifted to R. If $M(R)$ is commuting, then $M(R/N)$ is additive in $I(R/N)$ if and only if $M(R)$ is additive in $I(R)$.*

Theorem 2.10. *Let R be a ring and $e, f \in R$ be idempotents such that ef is a nonzero idempotent. Then we have the following:*

(1) *If $e \in M(R)$, then $eR = efeR = efR$ and $efe, ef \in M(R)$;*
(2) *If $f \in M(R)$, then $Rf = Rfef = Ref$ and $fef, ef \in M(R)$.*

Remark 3. In [3, Theorem 3.4], it was shown that if e, f are two nonzero primitive idempotents of a ring R such that $ef \neq 0$ is an idempotent, then e and f are conjugates. By using Theorem 2.10, we have alternative proof of [3, Theorem 3.4] as follows:

Since $e \in M(R)$ (resp. $f \in M(R)$) and $ef \neq 0, eR = efR$ (resp. $Rf = Ref$) by Theorem 2.10, and so ef and e are conjugates (resp. ef and f are conjugates). Hence e and f are conjugates.

Corollary 2.11. *Let R be a ring and $e, f \in R$ be idempotents such that fe is a nonzero idempotent. (1) If $e \in M(R)$, then $eR = fefR = feR$ and $fef, fe \in M(R)$; (2) If $f \in M(R)$, then $Rf = Refe = Rfe$ and $fef, fe \in M(R)$.*

Corollary 2.12. *Let R be a ring and $e, f \in R$ be idempotents. If e or f is central and $e \in M(R)$, then $eR = efR$ and $ef \in M(R)$.*

Corollary 2.13. *Let R be a ring. If $I(R)$ is multiplicative, then $M(R)$ is multiplicative.*

3. Some rings having multiplicative or additive set of idempotents

The following theorem was shown by Grover, Khurana, and Singh (see [3, Theorem 2.3]).

Theorem 3.1. *A ring R is a finite direct product of connected rings if and only if $M(R)$ is multiplicative and R has a complete set of primitive idempotents.*

By using the results obtained in Section 2, we have the following:

Theorem 3.2. *Let R be a ring with a complete set of primitive idempotents. Then the following are equivalent:*

(1) *$I(R)$ is multiplicative;*
(2) *R is a finite direct product of connected rings;*
(3) *$M(R)$ is commuting;*
(4) *$M(R)$ is multiplicative;*
(5) *$M(R)$ is additive in $I(R)$.*

Corollary 3.3. *Let R be a ring with a complete set of primitive idempotents. Then the following are equivalent:*

(1) *$I(R)$ is additive;*
(2) *R is a finite direct product of connected rings of characteristic 2;*
(3) *$M(R)$ is commuting and char(R) = 2;*
(4) *$M(R)$ is multiplicative and char(R) = 2;*
(5) *$M(R)$ is additive in $I(R)$ and char(R) = 2.*

Remark 4. Let R be a semiperfect ring. Then we can note that $I(R)$ is additive and $G = \{1\}$ if and only if $R \simeq \prod \mathbb{Z}_2$.

In [2, Proposition 9.9], it was shown that if R is a regular, right self-injective ring, then $B(R)$, the set of all central idempotents in R, is a complete Boolean algebra in which $e \wedge f = ef$ and $e \vee f = e + f - ef$ for all $e, f \in B(R)$. Note that if R is a von Neumann regular, right self-injective ring, then $1 = \vee B_o(R)$, the supremum of $B_o(R)$, where $B_o(R)$ is the set of orthogonal idempotents of $B(R)$ if and only if $R \simeq \prod_{e_i \in B_o(R)} e_i R$.

We may raise a natural question: Is it possible to extend Theorem 3.1 and Theorem 3.2 to the case of a direct product of countably many connected rings? In other words, assume that a ring R has a countably infinite set of pairwise orthogonal primitive idempotents, say $B = \{e_1, e_2, \dots\}$ such that $1 = \vee B$. Then is the condition that R *is a direct product of countably many connected rings* equivalent to the condition that $M(R)$ *is multiplicative*? But this does not hold true as the following example shows.

Example 1. Let K be a field and F be a proper subfield of K. Consider

$$R = \left\{ (a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} K \mid a_n \in F \text{ is eventurally} \right\},$$

which is a subring of $\prod_{n=1}^{\infty} K$. Then R is a von Neumann regular ring, and $Q(R) = \prod_{n=1}^{\infty} K$ is the maximal ring of quotients of R. Consider

$$B = \{e_1, e_2, \dots\} \subseteq Q(R),$$

where $e_1 = (1, 0, 0, \dots), e_2 = (0, 1, 0, \dots), \dots,$ and so on. Then B is a set of orthogonal primitive central idempotents in R. Further, $1 = \vee B$ in $I(R)$ because $Q(R) = \prod_{n=1}^{\infty} e_i Q(R)$ and $I(R) = I(Q(R))$. Also obviously, $M(R)$ is commuting.

On the other hand, assume that $R = \prod_{\lambda \in \Lambda} R_\lambda$, a countably infinite direct product of connected rings. Note that each R_λ is a commutative von Neumann regular connected ring. Thus each R_λ is a field, and so R is self-injective, which implies that $R = Q(R)$, a contradiction.

Recall that a central idempotent c of a ring R is said to be *centrally primitive* in R if $c \neq 0$ and c cannot be written as a sum of two nonzero orthogonal central idempotents in R (equivalently, cR is indecomposable as a ring).

Also, R is said to have a complete set of centrally primitive idempotents if there exists a finite set of centrally primitive pairwise orthogonal idempotents whose sum is 1 [4, Sects. 21 and 22].

Note that if a ring R has a complete set of primitive idempotents, then R has a complete set of centrally primitive idempotents.

We call a nonzero idempotent e in a ring R *fully basic* if e can be expressed as a sum of orthogonal primitive idempotents in R, and we call a ring R a *fully basic ring* if all idempotents in R are fully basic.

Examples:

(1) a finite direct product of local rings

(2) a ring of all upper triangular 2×2 matrices over $\mathbb{Z}_2$

Note that in a fully basic ring, $I(R)$ may not be multiplicative.

Theorem 3.4. *Suppose that a ring R has a complete set of primitive idempotents. If $I(R)$ is multiplicative, then R is a fully basic ring.*

Corollary 3.5. *A commutative semiperfect ring is a fully basic ring.*

Remark 5. Let $S = \{e_1, e_2, \ldots, e_r\}$ be a complete set of primitive idempotents. Then by Theorems 3.1,

(1) $I(R)$ is multiplicative if and only if $R \simeq R_1 \oplus R_2 \oplus \cdots \oplus R_r$ where all R_i's are connected rings, and then by Theorem 3.4, R is a fully basic ring.

(2) If $I(R)$ is multiplicative, then the number of all idempotents in R is equal to 2^r.

Remark 6. Let $S_k = \{e_{i_1}+e_{i_2}+\cdots+e_{i_k} : e_{i_1}, e_{i_2}, \ldots, e_{i_k} \in S, i_1 < i_2 < \cdots < i_k\}$ for each $k = 1, 2, \ldots, r$.

Then we have that

(1) $I(R) \cup \{1\} = \{0\} \cup S_1 \cup \cdots \cup S_r$.

(2) $|S_k| = {}_rC_k = \frac{r(r-1)\cdots(r-k+1)}{k(k-1)\cdots 1}$ for all $k = 1, \cdots, r$.

(3) $2^r = 1 + |S_1| + \cdots + |S_r| = 1 + {}_rC_1 + \cdots + {}_rC_r$.

References

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[4] T. Y. Lam, *A First Course in Noncommutative Rings*, Springer-Verlag, New York,, 1991.

Thank You Very Much !!