PARTIAL ACTIONS OF GROUPS ON SEMIPRIME RINGS

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Partial Actions

Let G be a group and R a unital k-algebra, k a commutative ring.

A **partial action of G on R** is a collection of ideals S_g, $g \in G$, of R, and isomorphisms $\alpha_g : S_{g^{-1}} \to S_g$ such that:

(i) $S_1 = R$ and α_1 is the identity mapping of R;

(ii) $S_{(gh)^{-1}} \supseteq \alpha_h^{-1}(S_h \cap S_{g^{-1}})$;

(iii) $\alpha_g \circ \alpha_h(x) = \alpha_{gh}(x)$, for any $x \in \alpha_h^{-1}(S_h \cap S_{g^{-1}})$.

The property (ii) easily implies that

$\alpha_g(S_{g^{-1}} \cap S_h) = S_g \cap S_{hg}$. for all $g, h \in G$.

Also $\alpha_{g^{-1}} = \alpha_g^{-1}$, for all $g \in G$.

This definition and the main notions on this subject have been given by M. Dokuchaev and R. Exel in the first paper on a pure algebraic context on partial actions published in 2005.

Partial skew group ring

Let α be a partial action of G on R. The **partial skew group ring** $R \star_\alpha G$

is defined as the set of all finite formal sums

$$\sum_{g \in G} a_g u_g, a_g \in S_g, \ \forall g \in G.$$

The addition is defined as in the usual way and the multiplication by

$$(a_g u_g)(b_h u_h) = \alpha_g(\alpha_{g^{-1}}(a_g) b_h) u_{gh}.$$

This algebra is not necessarily associative. There is an example in the paper by Dokuchaev and Exel of a partial action of a cyclic group of order two in an algebra such that the partial skew group ring is not associative.

It is an interesting question to find conditions under which it is associative. This is the case, for example, if R is a semiprime rings. There is a paper studying this question on modular group algebras.

Examples of partial actions

Assume that $\beta = \{\beta_g \mid g \in G\}$ is a global action of G on a k-algebra T. Let R be an ideal of T and suppose that R is not β-invariant. In this case the restriction of β to R is not a well-defined global action of G on R.

Anyway we can define a restriction as a partial action in the following way:

Take $S_g = R \cap \beta_g(R)$, for any $g \in G$, and define

$\alpha_g : S_{g^{-1}} \to S_g$ by restriction of β_g, i.e.,

$\alpha_g(x) = \beta_g(x)$, for any $x \in S_{g^{-1}}$ and $g \in G$.

It is not difficult to check that α is a well-defined partial action of G on R. This partial action is said to be the **restriction of β to a partial action on R.**

Enveloping Actions

Assume that $\alpha = \{S_g, \alpha_g : g \in G\}$ and $\alpha' = \{S'_g, \alpha'_g : g \in G\}$ are two partial actions of G on k-algebras R and R'.

We say that α and α' are **equivalent** if there exists an isomorphism of k-algebras $\phi : R \to R'$ such that for every $g \in G$ the following conditions are satisfied:

(i) $\phi(S_g) = S'_g$;

(ii) $\alpha'_g \circ \phi(x) = \phi \circ \alpha_g(x)$, for all $x \in S_{g^{-1}}$.

Definition Given a partial action α of G on R, a global action (T, β) is said to be an **enveloping action** of α if there exists an isomorphism ψ from R onto an ideal $\psi(R)$ of T such that α is equivalent to the partial action defined by restriction of the global action β to $\psi(R)$, and also T is generated by $\bigcup_{g \in G} \beta_g(\psi(R))$.

The additional condition above means that T is minimal amongst all the extensions satisfying the condition of the definition. As a consequence there is a more close connection between R and T. Also it implies the uniqueness of the enveloping action when it does exist.

The above definition can be given if the following way:

There exists a monomorphism ψ from R to T such that the following conditions are satisfied:

(i) $\psi(R)$ is an ideal of T;

(ii) $\psi(S_g) = \psi(R) \cap \beta_g(\psi(R))$, for any $g \in G$;

(iii) $\psi \circ \alpha_g(x) = \beta_g \circ \psi(x)$, for any $x \in S_{g^{-1}}$, $g \in G$;

(iv) T is equal to $\sum_{g \in G} \beta_g(\psi(R))$.

Hence, when α is a partial action of G on R and (T, β) is an enveloping action then, without loss of generality, unless equivalence, one can assume that the following conditions hold:

- R is an ideal of T;
- $S_g = R \cap \beta_g(R)$, for any $g \in G$;
- the restriction of β to R is equal to α;
- $T = \sum_{g \in G} \beta_g(R)$.

In the above situation, since for any $g \in G$ we have that

$$\beta_g(R) \simeq R,$$

there are closed relations between R and T.

In particular, when R is a semiprime ring, then T is also semiprime.

The question of finding necessary and sufficient conditions for a partial action to have an enveloping action have been obtained M. Dokuchaev and R. Exel, for rings with identity element.

They proved:

Theorem *Let R be a ring with an identity element and α a partial action of G on R. Then α has an enveloping action if and only if any of the ideals S_g, $g \in G$, has an identity element.*
Moreover, when the enveloping action exists, it is unique, unless equivalence.

The necessary and sufficient condition given above is equivalent to the condition: any of the ideals S_g is generated by a central idempotent of R. Denote the identity of R by 1_R and the identity of S_g by 1_g. It is easy to see that $1_g = 1_R\beta_g(1_R)$, where β_g is the corresponding automorphism of T.

Weak Enveloping Action

In the rest we assume that R does not have necessarily an identity

Definition *Given a partial action α of G on R, a* **weak enveloping action** *is a pair (T, β) such that T is a ring containing R, β is a global action of G on T such that for any $g \in G$ the mapping α_g is equal to the restriction of β to $S_{g^{-1}}$.*

The following has been proved (M. Ferrero):

Theorem *Let R be a semiprime ring and α a partial action of G on R. Then α has a weak enveloping action.*

The proof of the Theorem above is done as follows.

Let Q be the right (left or symmetric) Martindale ring of quotients of R. Given an ideal I of R the closure of I in R is defined as:

$$[I] = \{x \in R : \exists\ H \lhd_e R \text{ such that } xH \subseteq I\},$$

where $\lhd_e R$ denotes an essential ideal of R.

An ideal I of R is said to be closed if $[I] = I$.

Similarly, an extension I^* of I to Q is defined by

$$I^* = \{y \in Q : \exists\ H \lhd_e R \text{ such that } yH \subseteq I\}.$$

It is not difficult to show that I^* is always closed in Q and $I^* \cap R = [I]$.

It is well-known that in this case there exists a central idempotent $e \in C$ such that $I^* = eQ$, where C is the extended centroid of R, i.e., the center of Q.

For any $g \in G$ consider the ideal S_g^* of Q. The isomorphism $\alpha_g : S_{g^{-1}} \to S_g$ can be extended to an isomorphism $\alpha_g^* : S_{g^{-1}}^* \to S_g^*$.

It is not difficult to show that (S_g^*, α_g^*) defines a partial action α^* on Q extending α.

Since any ideal S_g^* is generated by a central idempotent, by Dokuchaev and Exel Theorem the partial action α^* has an enveloping action (T, β). So this is a weak enveloping action of α.

Since β extends α it is easy to see that the natural mapping

$$\Gamma : R \star_{\alpha} G \to T \star_{\beta} G, \ a_g u_g \mapsto a_g u_g,$$

is an inclusion. Since $T \star_{\beta} G$ is always associative we obtain

Corollary *If R is a semiprime ring and α is a partial action of G on R, then $R \star_{\alpha} G$ is an associative algebra.*

(Result already known by Dokuchaev-Exel paper- another proof)

Enveloping actions again

Now we discuss when a partial action of a group on a semiprime ring R (not necessarily with identity) has an enveloping action.

Given a partial action α of G on R we denote by α^* the extension of α to the Martindale ring of quotients Q, and by (T, β) the enveloping action of α^*.

Note that if $W = \sum_{g \in G} \beta_g(R)$, then W is β-invariant. Thus $(W, \beta|_W)$ is also a weak enveloping action, but it is closer to R.

The question is: find conditions under which $(W, \beta|_W)$ is an enveloping action of α.

The results in the following where obtained by the student L. Bemm and it will be contained in his PhD thesis. He begins his studies from results by W. Cortes and M. Ferrero:

Theorem A. *Assume that all the ideals S_g are closed. Then α has an enveloping action if and only if for any $a \in R$ and any $g \in G$ there exists a multiplier $\gamma_g(a)$ of R such that*:

(i) $R\gamma_g(a) \subseteq S_g$;

(ii) $\alpha_g(\alpha_{g^{-1}}(x)a) = x\gamma_g(a)$, *for any* $x \in S_g$.

Theorem B. *Assume that α has an enveloping action. Then α has an enveloping action which is semiprime. Moreover, in this case the enveloping action is equivalent to $(W, \beta|_W)$.*

Remark. When R has an identity element, then α has an enveloping action if and only if all the ideals S_g are generated by central idempotents of R. In this case all the ideals S_g are closed and the enveloping action is $(W, \beta|_W)$, unless equivalence.

In the following we list some results by L. Bemm.

Result 1. *If R has an identity element, then $(W, \beta|_W)$ is an enveloping action of α if and only if all the ideals S_g are closed and R is an ideal of W.*

Result 2. *Assume that all the ideals S_g are closed. Then $(W, \beta|_W)$ is an enveloping action of α if and only if R is an ideal of W.*

Result 3. *Assume that S_g is a direct summand of R, for any $g \in G$. Then α has an enveloping action which is equivalent to $(W, \beta|_W)$.*

Finally, examples are given to show the following:

Example 1. Even when R is an ideal of W, $(W, \beta|_W)$ is not necessarily an enveloping action of α if there exists an ideal S_g which is not closed in R.

Example 2. Even when the conditions (i) and (ii) of Theorem A are satisfied α has not necessarily an enveloping action if there exists an ideal S_g which is not closed.

Example 3. The converse of Result 3 does not hold: Even when all the ideals S_g are closed and $(W, \beta|_W)$ is an enveloping action of α not all the ideals S_g are necessarily direct summand of R.

Main References

Dokuchaev M. Exel, R.; Associativity of closed products by partial actions, enveloping actions and partial representations; Trans. AMS. 357, n. 5, (2005), 1931-1952.

W. Cortes, M. Ferrero; Globalizations of partial actions on semiprime rings; Contemp. Math. 499 (2009), 27-35.

M. Ferrero; Partial actions of groups on semiprime rings; Groups, Rings and Group Rings; Lect. Notes Pure and Appl. Math. 248; Chapman and Hall (2006), 155-162.

M. Ferrero, J. Lazzarin; Partial actions and partial skew group rings; J. Algebra 319 (2008), 5247, 5264.