Kernel Inclusions of Algebraic Automorphisms

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Outline

- Let R denote a prime ring.
- Q: symmetric Martindale quotient ring of R.
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- d: a derivation of R, namely,
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  $d(xy) = xd(y) + d(x)y$ for all $x, y \in R$.
- For example, for $a \in R$, $\mathrm{ad}(a) : x \mapsto [a, x] \stackrel{\text{def.}}{=} ax - xa$ is a derivation.
- For a derivation d of R, $\mathrm{Ker}(d) = \{x \in R \mid d(x) = 0\}$.
- Analogously, for an automorphism σ of R, $R^{(\sigma)} = \{x \in R \mid \sigma(x) = x\}$.
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Theorem (Brešar, 2005)

Assume char $R = 0$*. Let d and d' be derivations with d C-algebraic. Assume* $\mathrm{Ker}(d) \subseteq \mathrm{Ker}(d')$*. Then* $d = \mathrm{ad}(a)$ *and* $d' = \mathrm{ad}(p(a))$ *for some $a \in Q$ and $p(x) \in C[X]$.*

In particular, $d' = \alpha_n d^n + \alpha_{n-1} d^{n-1} + \cdots + \alpha_1 d$ for some $\alpha_i \in C$.

Theorem (Chuang, Lee, Wong, 2008)

Assume char $R = p$*. Let d and d' be derivations with d C-algebraic. Assume* $\mathrm{Ker}(d) \subseteq \mathrm{Ker}(d')$*. Then* $d' = \sum \alpha_i d^{p^i} + \mathrm{ad}(b)$ *for some $b \in Q$.*

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Motivated by these results, we concern kernel inclusion problem for automorphisms. We have the following:

Theorem

Let σ and τ be automorphisms of R and assume that σ is C-algebraic. Then $R^{(\sigma)} \subseteq R^{(\tau)}$ if and only if $\tau(x) = v\sigma^i(x)v^{-1}$ for all $x \in R$, where i is an integer and where v is in the centralizer of $R^{(\sigma)}$ in Q.

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A set of automorphisms of R, say $\{\sigma_1, \sigma_2, \ldots, \sigma_m\}$, is called mutually outer if $\sigma_i\sigma_j^{-1}$ is X-outer for all $i \neq j$.

Theorem (Kharchenko, 1975)

Let $\sigma_1, \sigma_2, \ldots, \sigma_m$ be mutually outer automorphisms of R. For $a_i, b_i, c_j, d_j \in Q$, suppose that

$$\sum_{i=1}^{m} \sum_{j=1}^{n_i} a_{ij}\sigma_i(x)b_{ij} = 0$$

for all $x \in R$. Then

$$\sum_{j=1}^{n_i} a_{ij}yb_{ij} = 0$$

for all $y \in R$, for $i = 1, 2, \ldots, m$.

Example

- Assume $\sigma_1, \sigma_2, \sigma_3$ are mutually outer and

$$\begin{array}{ccccccc} a\sigma_1(x)b & + & c\sigma_1(x)d & + & e\sigma_1(x)f & & \\ & + & g\sigma_2(x)h & + & i\sigma_2(x)j & & \\ & & & + & k\sigma_3(x)\ell & = & 0. \end{array}$$

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- $\sigma_1, \sigma_2, \sigma_3$ are automorphisms and hence onto. Substitute x with $\sigma_1^{-1}(y)$, we get $ayb + cyd + eyf = 0$ for all $y \in R$. Analogously, $gyh + iyj = 0$ and $ky\ell = 0$ for all $y \in R$.

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Theorem (Martindale, 1969)

Let $a_i, b_i \in Q$ for $1 \leq i \leq m$. Assume that

$$\sum_{i=1}^{m} a_i x b_i = 0$$

for all $x \in R$. If $a_1, \ldots, a_m$ are C-independent, then $b_i = 0$.

- The spirit of Kharchenko's Theorem and Martindale's Theorem is to reduce the GPI (with automorphisms) as possible as we can.
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- Collect the terms depending on the automorphism acting on x. Say $\sigma_1, \cdots, \sigma_m$.
- $\sum_{i=1}^{m} \sum_{j=1}^{n_i} a_{ij}\sigma_i(x)b_{ij} = 0$.
- If $\sigma_1, \cdots, \sigma_m$ are not mutually outer, then $\sigma_i\sigma_j^{-1}$ is X-inner for some $i \neq j$. Say $\sigma_i\sigma_j^{-1} = \iota(v)$. Then $\sigma_i = \iota(v)\sigma_j$, namely $\sigma_i(x) = v\sigma_j(x)v^{-1}$.
- Substitute $\sigma_i(x)$ with $v\sigma_j(x)v^{-1}$ in the current identity.
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- Collect the terms depending on the left coefficient. Say the left coefficients are $a_1, \cdots, a_m$.
- $\sum_{i=1}^{m} a_i x b_i = 0$.
- Assume $a_1, \cdots, a_m$ are not C-independent. Say $a_1 = \alpha_2 a_2 + \cdots + \alpha_m a_m$.
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Theorem

Let σ and τ be automorphisms of R and assume that σ is C-algebraic. Then $R^{(\sigma)} \subseteq R^{(\tau)}$ if and only if $\tau(x) = v\sigma^i(x)v^{-1}$ for all $x \in R$, where i is an integer and where v is in the centralizer of $R^{(\sigma)}$ in Q.

Sketch of Proof

The implication "$\Rightarrow$" is trivial. So we assume $R^{(\sigma)} \subseteq R^{(\tau)}$.

- *σ is algebraic and then (by Chuang and Lee's method) one construct a polynomial expression $\psi(x) \in R^{(\sigma)}$ for all $x \in J \lhd R$.*
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$R^{(\sigma)} \subseteq R^{(\tau)}$. *$\sigma$ is X-outer and $\sigma^2 = \iota(1) = I$.*

Proof.

1. For all $x \in R$, let $\psi(x) = x + \sigma(x)$.
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4. If $\tau, \tau\sigma, \tau\sigma^2$ are all X-outer, then $I, \sigma, \sigma^2, \tau, \tau\sigma, \tau\sigma^2$ are mutually outer.
5. By Kharchenko's Theorem, $1 \cdot X \cdot 1 + u \cdot X \cdot (u^{-1} + 1) = 0$.
6. By Martindale' Theorem, $1 = 0$, $\rightarrow\leftarrow$.

Example 2

$R^{(\sigma)} \subseteq R^{(\tau)}$. *$\sigma, \sigma^2$ are X-outer and $\sigma^3 = \iota(u)$, $u^2 + u + 1 = 0$.*

Proof.

1. $\sigma(u) = u$ by minimality of the algebraic relation.
2. By Chuang and Lee's construction,
$$\begin{aligned} \psi(x) = {} & x + uxu^{-1} + ux \\ & + \sigma(x) + u\sigma(x)u^{-1} + u\sigma(x) \\ & + \sigma^2(x) + u\sigma^2(x)u^{-1} + u\sigma^2(x) \end{aligned}$$
3. $\sigma(\psi(x)) = \psi(x)$ and then $\tau(\psi(x)) = \psi(x)$.
4. If $\tau, \tau\sigma, \tau\sigma^2$ are all X-outer, then $I, \sigma, \sigma^2, \tau, \tau\sigma, \tau\sigma^2$ are mutually outer.
5. By Kharchenko's Theorem, $1 \cdot X \cdot 1 + u \cdot X \cdot (u^{-1} + 1) = 0$.
6. By Martindale' Theorem, $1 = 0$, $\rightarrow\leftarrow$.

$\square$

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