Decompositions of Quotient Rings and m-Commuting Maps

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- C : extended centroid of R.

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- $S_k(X_1, \ldots, X_k) := \sum_{\sigma \in Sym(k)} (-1)^\sigma X_{\sigma(1)} \cdots X_{\sigma(k)}$.

  For example, $S_2(X_1, X_2) = X_1X_2 - X_2X_1$.

Theorem 1 (T. Kosan, T.-K. Lee, Y. Zhou)

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1. Q_1 *is a ring satisfying* f.
2. Q_2 *is a faithful* f*-free ring.*

Theorem 2 (C. -W. Chen and T. -K. Lee, 2011)

Let $n \geq 2$ and $f(X) = X^n h(X)$ where $h(X)$ is a polynomial over $\mathbb{Z}$ with $h(0) = \pm 1$. Then there is a decomposition

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such that the following hold:

1. Q_1 *is a ring satisfying* S_{2n-2}.
2. Q_2 *satisfies* f *and* $Q_2 \cong M_n(E)$ *where* E *is a commutative self-injective ring such that, for some fixed integer* $q > 1$, $x^q = x$ *for all* $x \in E$.

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3. Q_3 *is a both faithful* S_{2n-2}*-free and faithful* f*-free ring.*

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- A map by linear generalized differential polynomial:

$$X \to \sum_i \sum_j a_{ij} X^{\Delta_j} b_{ij}$$

where $a_{ij}, b_{ij} \in Q$ and derivation words Δ_j.

Theorem 5 (T.-K. Lee, K.-S. Liu, W.-K. Shiue, 2004)

Let R be a noncommutative prime ring. $R \ncong M_2(GF(2))$. $\psi : Q \to Q$ satisfying $[\psi(x), x^m] = 0$ for all $x \in R$, where ψ defined by a linear generalized differential polynomial. Then

$$\psi(x) = \lambda x + \mu(x)$$

for all $x \in R$, where $\lambda \in C$ and $\mu : R \to C$.

Lemma 6 (C.-W. Chen and T.-K. Lee, 2011)

Let

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1. *R is both faithful f-free and faithful S_2-free.*
2. *For any minimal prime ideal P of Q, neither Q/P is commutative, nor $Q/P \cong M_2(GF(2))$.*

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Theorem 8 (C.-W. Chen and T.-K. Lee, 2011)

Suppose that $\psi : Q \to Q$ *satisfying* $[\psi(x), x^m] = 0$ *for all* $x \in R$*, where* ψ *defined by a linear generalized differential polynomial. Then*

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3. $\psi(x) = \lambda x + \mu(x)$ *for all* $x \in Q_3$*, where* $\lambda \in C$ *and* $\mu : Q_3 \to C$*.*

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$R := \mathrm{M}_2(\mathrm{GF}(2))$. $f : R \to R$ defined by

$$f\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) = \begin{pmatrix} \alpha+\gamma & 0 \\ 0 & \beta+\delta \end{pmatrix}$$

for $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in R$.

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1. $[f(x), x^6] = 0$ for all $x \in R$.
2. $\left[f\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right), \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.