On Fully-M-cyclic modules.

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1. Introduction

Definition 1.1. (From N. V. Sanh , K. P. Shum , S. Dhompongsa and S. Wongwai: On quasi-principally injective modules, Algebra Colloquium, **6**(3), 269-276(1999).)

Let M be a right R-module and $S = End_R(M)$. A right R-module N is said to be *M-p-injective* if for any $s \in S$, and every homomorphism from $s(M)$ to N can be extended to a homomorphism from M to N.

Definition 1.2. (From S. Baupradist,H. D. Hai and N. V. Sanh, On pseudo-p-injectivity, to appear at *Southeast Asian Bull. Math.*)

Let M be a right R-module and $S = End_R(M)$. A right R-module N is said to be *pseudo-M-p-injective* (resp. M-p-injective) if for any $s \in S$, and every monomorphism (resp. homomorphism) from $s(M)$ to N can be extended to a homomorphism from M to N.

Definition 1.3. (From S. Baupradist,H. D. Hai and N. V. Sanh, A General form of Pseudo-p-Injectivity, to appear at *Southeast Asian Bull. Math.*)

Let M be a right R-module and $S = End_R(M)$. A right R-module N is said to be *generalized-pseudo-M-p-injective*, if for any $0 \neq s \in S$, there is an $n \in \mathbb{N}$ such that $s^n \neq 0$ and any monomorphism from $s^n(M)$ to N can be extended to a homomorphism from M to N.

The aim of this work was to generalize generator, M-generated modules in order to apply them to a wider class of rings and modules. We started by establishing a new concept which is called a fully-M-cyclic module. We defined this notation by using $Hom_R(M, *)$ operators which are helpful to contract the new construction and describe their properties. Finally, we could see the structure of fully-M-cyclic module and quasi-fully-cyclic module by the structure of M.

2. On Fully-M-cyclic module.

In this part, a module M be given as a right R-module.

Definition 2.1. Let $N \in M_R$. N is called a fully-M-cyclic module if every submodule A of N is of the form $s(M)$ for some s in $Hom_R(M, N)$.

Remark 2.2. Dealing directly from definition, the following statements are routine:
(1) Submodule of a fully-M-cyclic module is a fully-M-cyclic module.
(2) If M is simple module and N is fully-M-cyclic module, then any nonzero submodule of N is simple submodule.

Definition 2.3. The module $M \in M_R$ is called a quasi-fully-cyclic module if it is a fully-M-cyclic module.

Obviously, every semi-simple module is a quasi-fully-cyclic module.

Lemma 2.4. *Let N be a fully-M-cyclic module. If M is a noetherian module then $Soc(M) \cong Soc(N)$.*

Lemma 2.5. *If N is a fully-M-cyclic module then N has no nonzero small submodule.*

Corollary 2.6. *If N is a fully-M-cyclic module then $Rad(N) = 0$.*

Definition 2.7. Let N be a fully-M-cyclic module. For a submodule A of N there exists a homomorphism $s \in Hom_R(M, N)$ such that $s(M) = A$. s is called a *presented homomorphism* of A.

Lemma 2.8. *Let N be a fully-M-cyclic module. If s is a presented homomorphism of a submodule A of N then A is maximal if and only if every $t \in S = Hom_R(M, N)$ with $Im(t)$ containing the image of presented homomorphism of A is an epimorphism.*

Leading directly from definition, the following properties in Lemma 2.9 are routine,

Lemma 2.9. *Let N be a fully-M-cyclic module and A be a submodule of N and s its a presented homomorphism.*
(1) *If M is an epimorphism image of M' then N is also a fully-M'-cyclic module.*
(2) *If M is a fully-M'-cyclic module then N is also a fully-M'-cyclic module.*

(3) *A is an essential in N if and only if for any nonzero element t of $Hom_R(M, N)$, $Im(t) \cap Im(s) \neq 0$.*
(4) *A is uniform if and only if every $t \in Hom_R(M, N)$ with $0 \neq Im(t) \subset_{>} Im(s)$ then $Im(t)$ is an essential in $Im(s)$.*
(5) *A is a direct summand of N if and only if there exists $t \in Hom_R(M, N)$ such that $Im(s) \cap Im(t) = 0$ and $s + t$ is an epimorphism.*

3. Quasi-fully-cyclic module.

In this part, we put $S = End_R(M)$. We have known that for any right R-module M, the direct summand A of M is image of a presented homomorphism which is an idempotent of S but not all. Which is case of the form submodules such that every its presented homomorphisms are idempotents?. The following lemma is a clear answer:

Lemma 3.1. *Let M be a quasi-fully-cyclic module. If A is a simple submodule of M with s its a presented homomorphism then s is an idempotent of $S = End_R(M)$.*

Right now, we suppose that M be a quasi-fully-cyclic module. If $e^2 = e$, the one gets a direct sum decomposition $M = e(M) \oplus (1-e)(M)$. Conversely, if $M = A \oplus B$ then we can write $1 = \pi_A + \pi_B$ with π_A (resp. π_B) being a natural projection map from M to A (resp. B). π_A (resp. π_B) is an idempotent element of S which is a presented homomorphism of A (resp. B) so that we can get the following corollary.

Corollary 3.2. *In a quasi-fully-cyclic module, every simple submodule is a direct summand.*

Theorem 3.3. *Let M be a quasi-fully-cyclic module. M is a Noetherian (resp. Artinian) if and only if S is a right self Noetherian (resp. Artinian) ring.*

Lemma 3.4. *For each quasi-fully-cyclic-module, the following statements are equivalent:*
(1) *S is artinian;*
(2) *M is finitely co-generated;*
(3) *M is semisimple and finitely generated;*
(4) *M is semisimple and noetherian;*

(5) *M is the direct sum of a finite set of simple submodules.*

Definition 3.5. Let M be a right R-module. M is called *Hopfian* (resp. *co-Hopfian*) if every surjective (resp. injective) endomorphism of M is an automorphism.

Definition 3.6. Let M be a right R-module. M is called a *Fitting* module if every endomorphism f of M satisfies Fitting's lemma (i.e there exists an integer $n \geq 1$ such that $M = Ker(f^n) \oplus Im(f^n)$).

Lemma 3.7. *Let M be a quasi-fully-cyclic-module. If M is finitely cogenerated and Hopfian then for any $s \in S$ there exists an integer number n such that $M = Ker(s^n) \oplus Im(s^n)$.*

Theorem 3.8. *Let M be a quasi-fully-cyclic module.*
(1) *For any $s, u \in S, l_S(Im(u)) + Ss \subset_{>} l_S(Im(u) \cap Ker(s))$.*
(2) If N is a maximal submodule of M then $l_S(N)$ is a minimal left ideal of S.

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