ON A POSSIBLY NEW CONSTRUCTION IN ALGEBRAIC K-THEORY AND OPEN PROBLEMS CONCERNING IT

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This talk is based on the paper:

[1] M. Abel, "On Algebraic K-Theory" published in "*ICTAA 2008, Proceedings, Tartu 2008*", Mathematics Studies **4**, 2008, pp. 7–12.

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In case R is nonunital ring, the following construction is standard: Consider the unitization $\widetilde{R} := R \times \mathbb{Z}$ equipped with the multiplication $\tilde{\cdot}$ defined as $(p,m)\tilde{\cdot}(r,n) := (p \cdot r + np + mr, mn)$ for every $(p,m),(r,n) \in \widetilde{R}$.

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Define a map $f : \widetilde{R} \to \mathbb{Z}$ defined by $(r,z) \mapsto z$ for every $r \in R$ and $z \in Z$, which induces naturally a map $M_\infty(\widetilde{R}) \to M_\infty(\mathbb{Z})$.

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$$K_0(R,\cdot) = \ker\Big(K_0(\widetilde{R},\tilde{\cdot}) \to K_0(\mathbb{Z},\cdot)\Big),$$

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What about the formulae, which would not depend on the existence of a unit?

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Instead of the equivalence $\approx$ defined by $M \approx N$ if and only if there exists $P \in \mathrm{GL}^{\cdot}_\infty(R)$ such that $M = P^{-1}NP$, consider another equivalence $\approx_q$ defined by $M \approx_q N$ if and only if there exists $Q \in \mathrm{QGL}^{\circ}_\infty(R)$ such that $M = Q_q^{-1} \circ N \circ Q$.

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Instead of the set

$$[\mathrm{GL}^{\cdot}_\infty(R), \mathrm{GL}^{\cdot}_\infty(R)] = < M^{-1}N^{-1}MN \mid M, N \in \mathrm{GL}^{\cdot}_\infty(R) >$$

consider the set

$$[\mathrm{QGL}^{\circ}_\infty(R), \mathrm{QGL}^{\circ}_\infty(R)] = < M_q^{-1} \circ N_q^{-1} \circ M \circ N \mid M, N \in \mathrm{QGL}^{\circ}_\infty(R) > .$$

Suppose that R is an arbitrary (i.e. not necessarily unital) ring and define its "new" K-groups by

$$\overline{K_0}(R) := (\mathrm{QIdem}^{\circ}_{\infty}(R)/_{\approx_q}, \oplus)^{+},$$

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In [1] we showed that in case R is unital, we have

$$K_0(R,\cdot) \cong \overline{K_0}(R) \text{ and } K_1(R,\cdot) \cong \overline{K_1}(R).$$

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In order to show that the similar equivalence holds also for non-unital R, we have to show that

$$\ker\left((\mathrm{QIdem}_{\infty}^{\tilde{\circ}}(\widetilde{R})/_{\approx_q}, \oplus)^{+} \to (\mathrm{QIdem}_{\infty}^{\circ}(\mathbb{Z})/_{\approx_q}, \oplus)^{+}\right) \cong$$

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$$\ker\left(\frac{\mathrm{QGL}_{\infty}^{\tilde{\circ}}(\widetilde{R})}{[\mathrm{QGL}_{\infty}^{\tilde{\circ}}(\widetilde{R}), \mathrm{QGL}_{\infty}^{\tilde{\circ}}(\widetilde{R})]} \to \frac{\mathrm{QGL}_{\infty}^{\circ}(\mathbb{Z})}{[\mathrm{QGL}_{\infty}^{\circ}(\mathbb{Z}), \mathrm{QGL}_{\infty}^{\circ}(\mathbb{Z})]}\right) \cong$$

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In 2010 we were able to show that

$$\ker\left(\mathrm{QIdem}^{\tilde{\circ}}_{\infty}(\tilde{R})/_{\approx_q} \longrightarrow \mathrm{QIdem}^{\circ}_{\infty}(\mathbb{Z})/_{\approx_q}\right) \cong \mathrm{QIdem}^{\circ}_{\infty}(R)/_{\approx_q}.$$

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Open question: is it true that $K_0(R,\cdot) \cong \overline{K_0}(R,\cdot)$ for nonunital ring R?

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